

## THE CRITICAL POINT EQUATION ON A FOUR DIMENSIONAL WARPED PRODUCT MANIFOLD

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ABSTRACT. On a compact oriented  $n$ -dimensional manifold  $(M^n, g)$ , it has been conjectured that a metric  $g$  satisfying the critical point equation (2) should be Einstein. In this paper, we prove that if a manifold  $(M^4, g)$  is a 4-dimensional oriented compact warped product, then  $g$  can not be a solution of CPE with a non-zero solution function  $f$ .

### 1. Introduction

Let  $M$  be an  $n$ -dimensional compact orientable manifold. Denoting the space of all smooth Riemannian metrics on  $M$  by  $\mathcal{RM}$ , let  $\mathcal{M}$  be the quotient of  $\mathcal{RM}$  by the group of all diffeomorphisms of  $M$ . Then, for a given smooth structure  $g \in \mathcal{M}$ , its scalar curvature  $s_g$  is an element of the space of  $C^\infty(M)$  functions, and the linearization of the scalar curvature is given by

$$s'_g(h) = -\Delta_g \operatorname{tr} h + \delta_g^* \delta_g h - g(h, r_g),$$

where  $\Delta_g$  is the negative Laplacian of  $g$ ,  $r_g$  its Ricci tensor,  $\delta$  the divergence operator, and  $\delta^*$  is the formal adjoint of  $\delta$ . Also, the  $L^2$ -adjoint operator  $s_g'^*$  of  $s'_g$  is given by

$$(1) \quad s_g'^*(f) = -g\Delta_g f + D_g df - f r_g$$

and the critical point equation, denoted *CPE hereafter*, is given by ([1])

$$(2) \quad z_g = s_g'^*(f),$$

where  $z_g$  is the traceless Ricci tensor, and  $f$  a function on  $M^n$  with vanishing mean value.

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Received November 30, 2004. Revised April 22, 2006.

2000 Mathematics Subject Classification: 53C25.

Key words and phrases: critical point equation, warped product, Einstein metric.

\* This research was supported by Korea Research Foundation Grant (KRF-2003-003-C00014).

For the case that the function  $f$  in CPE is trivial, it is clear that the metric  $g$  is Einstein. Therefore, all considerations in this paper are restricted to a *non-trivial function  $f$  only*. For the case that  $f$  is a smooth non-trivial function, the following statement has been conjectured ([1]):

**CONJECTURE A.** *If CPE holds for a non-trivial function  $f$ , then the metric  $g$  of the manifold  $M$  is Einstein.*

If this conjecture is true, it is known that  $(M, g)$  is isometric to a standard sphere  $S^n$  [10]. It turns out to be difficult to solve Conjecture A even with additional assumptions imposed on the metric. However, there are some partial answers to Conjecture A, such as those in [4] and [9] for example.

The purpose of the present paper is to prove the following Main Theorem as a partial (negative) answer to Conjecture A:

**MAIN THEOREM.** *Let  $(M, g)$  be a 4-dimensional oriented compact warped product given by  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with  $\psi > 0$ . Then  $g$  can not be a solution of CPE with a non-zero solution function  $f$ .*

**REMARK 1.** Fisher and Marsden suggested the F-M conjecture in [3], stating that if a smooth function  $f$  satisfies  $s_g^*(f) = 0$ , then a solution metric  $g$  is isometric to the standard sphere. For the relationships between Conjecture A and F-M conjecture may be found in [5]. However, counter-examples of F-M conjecture were found (e.g., see [8]), mostly warped product manifolds. Therefore, it naturally arises a question to ask whether there exists a warped product metric which constitutes a counter-example of Conjecture A.

Contrary to the case of F-M conjecture, our Main Theorem shows that no 4-dimensional warped products satisfy CPE unless they are isometric to standard spheres. Combining this result with that of 3-dimensional case in [6], we may conclude that the answer to the question in the previous paragraph is no for  $n \leq 4$ .

It is natural to ask a similar question for  $n \geq 5$ . However, the difficulty for answering the question increases greatly, and is beyond our current understanding of the problem.

**REMARK 2.** For a 4-dimensional warped product manifold  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with  $\psi > 0$ , it is clear that  $B$  is complete if  $M$  is complete. If we consider the case when the 1-dimensional base space  $B$  is incomplete with  $\psi \geq 0$  and the fiber  $F$  is Einstein, we may conclude that if  $g$  is a solution metric of (3) with a non-zero solution function  $f$ ,

then  $(M, g)$  is isometric to the standard sphere  $S^4$  (Proposition 11). It will be shown in Section 5.

In the present paper, our Main Theorem will be proved as follows. For a 4-dimensional warped product manifold  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ , only the following 3 cases are possible with respect to the dimension of  $B$  and  $F$ :

Case 1.  $\dim B = 1, \dim F = 3$

Case 2.  $\dim B = 2, \dim F = 2$

Case 3.  $\dim B = 3, \dim F = 1$ .

For the case that  $B$  is complete, we prove a non-existence theorem (Main Theorem) in Section 3. For the case that  $B$  is not complete with  $\dim B = 1$ , a rigidity result is shown in Section 5.

## 2. Preliminaries

This section is a brief collection of notations and results, which are needed in our subsequent considerations.

Among the partial answers to the conjecture A given in section 1, the following three theorems, which hold on an  $n$ -dimensional manifold  $(M, g)$ , are needed in the next sections.

**THEOREM 1.** *Let  $g$  be a solution metric of CPE. If  $g$  is conformally flat, then  $(M, g)$  is isometric to a standard sphere [9].*

**THEOREM 2.** *Let  $(g, f)$  be a solution of CPE. If  $f$  always takes values greater than or equal to  $-1$ , then  $(M, g)$  is isometric to a standard sphere [4].*

**THEOREM 3.** *Let  $g$  be a solution metric of CPE. If the metric  $g$  is Einstein,  $(M, g)$  is isometric to a standard sphere  $S^n$  [10].*

Furthermore, using (1) and (2) we have another useful representation of CPE on an  $n$ -dimensional manifold  $(M, g)$ , which may be written as

$$(3) \quad (1 + f)z_g = D_g df + \frac{s_g f}{n(n-1)}g.$$

Taking the trace of (3), we have  $\Delta_g f = -\frac{s_g}{n-1}f$ . Note that the scalar curvature  $s_g$  of the metric  $g$  satisfying CPE is assumed to be constant ([1]). Therefore we have  $\int_M f = 0$ , and hence  $f$  takes both positive and negative values.

Finally, on a 4-dimensional compact warped product manifold  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$ , let

- $\check{r}, \check{s}$ : the lifts to  $M$  of Ricci and scalar curvature of  $B$ , respectively
- $\hat{r}, \hat{s}$ : the Ricci and scalar curvature of  $F$ , respectively
- $X_i$ : a lifted horizontal orthonormal frame field,  $i = 1, \dots, \dim B$
- $U_j$ : a lifted vertical orthonormal frame field,  $j = 1, \dots, \dim F$
- $\mathcal{V}$ : the vertical distribution.

Then, the following two propositions hold on  $M$  for each of 3 cases mentioned in the last paragraph of section 1:

PROPOSITION 4. We have ([1])

Case 1.  $\dim B = 1$  and  $\dim F = 3$ ;

$$\begin{aligned} r(X, X) &= -\frac{3\psi''}{\psi} \\ r(U_i, U_j) &= \hat{r}(U_i, U_j) + \langle U_i, U_j \rangle \left( -\frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2} \right) \\ s &= \frac{\hat{s}}{\psi^2} - 6 \left( \frac{\psi''}{\psi} + \frac{\psi'^2}{\psi^2} \right) \end{aligned}$$

Case 2.  $\dim B = 2$  and  $\dim F = 2$ ;

$$\begin{aligned} r(X_i, X_j) &= \check{r}(\check{X}_i, \check{X}_j) - \frac{2}{\psi} \check{D}d\psi(\check{X}_i, \check{X}_j) \\ r(U_i, U_j) &= \hat{r}(U_i, U_j) + \langle U_i, U_j \rangle \left( -\frac{\check{\Delta}\psi}{\psi} - \frac{|d\psi|^2}{\psi^2} \right) \\ s &= \check{s} + \frac{\hat{s}}{\psi^2} - 4\frac{\check{\Delta}\psi}{\psi} - 2\frac{|d\psi|^2}{\psi^2} \end{aligned}$$

Case 3.  $\dim B = 3$  and  $\dim F = 1$ ;

$$\begin{aligned} r(X_i, X_j) &= \check{r}(\check{X}_i, \check{X}_j) - \frac{1}{\psi} \check{D}d\psi(\check{X}_i, \check{X}_j) \\ r(U, U) &= -\frac{\check{\Delta}\psi}{\psi} \\ s &= \check{s} - 2\frac{\check{\Delta}\psi}{\psi}. \end{aligned}$$

The next corollary follows from Proposition 4.

COROLLARY 5. If the scalar curvature  $s$  of  $M$  is constant, then the scalar curvature  $\hat{s}$  of  $F$  is constant.

*Proof.* Proposition 4 gives  $\hat{s} = \psi^2 s + 6(\psi''\psi + \psi'^2)$  in the Case 1. Since  $\hat{s}$  is a function on  $F$ , and the right-hand side of this equation is a function on  $B$ ,  $\hat{s}$  should be a constant function on  $F$ , i.e.,  $F$  is of constant scalar curvature. In the Case 2, this corollary follows similarly, since we have  $\hat{s} = \psi^2(s - \check{s}) + 4\psi\check{\Delta}\psi + 2|d\psi|^2$ . Finally, we have  $\hat{s} = 0$  in the Case 3.  $\square$

### 3. Proof of main theorem

This section is devoted to the proof of our Main Theorem. Throughout this section we assume that  $B$  is complete with  $\psi > 0$ . The proof of Main Theorem follows directly from the following lemmas. For the Case 1 ( $\dim B = 1$ ), Lemma 6 and 7 give the proof. Lemma 8 and 9 give the proof of the Case 2 ( $\dim B = 2$ ). For the remaining Case 3 ( $\dim B = 3$ ), the proof follows from Lemma 10.

LEMMA 6. *Let  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with the 1-dimensional complete base  $B$ . Suppose that  $f$  is a function of  $B$  only. Then  $g$  can not be a solution of (3).*

*Proof.* First note that

$$(4) \quad \langle D_{U_i} df, U_i \rangle = \langle D_{U_i} \check{d}f, U_i \rangle = \frac{\psi' f'}{\psi},$$

where we used the fact that  $D_{U_i} \check{d}f = \frac{\psi' \partial_t f}{\psi} U_i$  [1]. Now, from (3) we have

$$(1+f) \left( \hat{r}(U_i, U_i) - \frac{\psi''}{\psi} - 2 \frac{\psi'^2}{\psi^2} - \frac{s}{4} \right) = \frac{\psi'}{\psi} f' + \frac{sf}{12}.$$

Then it is easy to see that  $\hat{r}(U_1, U_1) = \hat{r}(U_2, U_2) = \hat{r}(U_3, U_3)$ , or  $\hat{r}(\hat{U}_i, \hat{U}_i) = \frac{\hat{s}}{3}$  for all  $1 \leq i \leq 3$ , and  $\hat{r}(U_i, U_j) = 0$  for any  $i \neq j$ . Thus  $F$  is an Einstein manifold. Since  $F$  is 3-dimensional, it follows that  $F$  should be of constant sectional curvature. In other words,  $F$  is isometric to  $S^3/\Gamma$ , with  $\Gamma \subset SO(4)$ . Hence we have  $M = S^1 \times_{\psi^2} (S^3/\Gamma)$  with the metric  $g$  given by  $g = dt^2 + \psi(t)^2 g_0$ . This metric  $g$  is conformally flat, since  $g_0$  is of constant curvature, c.f. [8]. Then, by Theorem 1,  $(M, g)$  is isometric to a standard sphere  $S^4$ . This is clearly a contradiction, since there is no nonvanishing function  $\psi$  with  $S^4 = S^1 \times_{\psi^2} (S^3/\Gamma)$ .  $\square$

LEMMA 7. *Let  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with the 1-dimensional complete base  $B$ . Then  $g$  can not be a solution of (3).*

*Proof.* From (3), we have

$$(5) \quad 0 = (1+f)z(X, U_i) = \langle D_X df, U_i \rangle = X \langle df, U_i \rangle - \langle df, D_X U_i \rangle$$

for a lifted horizontal vector field  $X$ . Note that  $U_i = \frac{1}{\psi} \hat{U}_i$ , where  $\hat{U}_i$  is a lift of vector field on  $F$ . Therefore

$$(6) \quad D_X U_i = D_X \left( \frac{1}{\psi} \hat{U}_i \right) = -\frac{X(\psi)}{\psi^2} \hat{U}_i + \frac{1}{\psi} D_X \hat{U}_i = 0,$$

where we used the fact that  $D_X \widehat{U}_i = \frac{X(\psi)}{\psi} \widehat{U}_i$  [11] in the last equality. Now, substitution of (6) into (5) gives

$$(7) \quad XU_i(f) = X\langle df, U_i \rangle = 0.$$

Therefore  $U_i(f) = \frac{1}{\psi} \widehat{U}_i(f)$  is a function of  $F$  only, and it is easy to see that  $f$  can be written as

$$(8) \quad f = \psi b + c,$$

where  $b$  is a function on  $F$  and  $c = c(t)$  is a function on  $B$ . Substituting (8) into  $(1+f)z(X, X) = \langle D_X df, X \rangle + \frac{s}{12}$ , we have

$$(9) \quad (1 + \psi b + c)(-3\frac{\psi''}{\psi} - \frac{s}{4}) = \psi''b + c'' + \frac{s}{12}(\psi b + c),$$

where we used the fact that, from  $\check{d}f = b d\psi + dc$  and  $D_X \widehat{d}f = \mathcal{V} D_X \widehat{d}f$ ,

$$\begin{aligned} \langle D_X df, X \rangle &= \langle D_X \check{d}f, X \rangle + \langle D_X \widehat{d}f, X \rangle = \langle D_X \check{d}f, X \rangle \\ &= b \langle D_X d\psi, X \rangle + \langle D_X dc, X \rangle = \psi''b + c''. \end{aligned}$$

Thus, the equation (9) can be rewritten as

$$(10) \quad b(-4\psi'' - \frac{s}{3}\psi) = (1+c)(\frac{3\psi''}{\psi} + \frac{s}{4}) + c'' + \frac{s}{12}c.$$

Note that both  $-4\psi'' - \frac{s}{3}\psi$  and the right-hand side are functions of  $B$  only, while  $b$  is a function of  $F$ . Thus, in order that the equation (10) holds for any  $t$ , either  $b$  is constant or  $-4\psi'' - \frac{s}{3}\psi$  has to be zero. If  $b$  is constant,  $f$  is a function of  $B$  only, and so  $g$  can not be a solution of (3) in virtue of Lemma 6. Now, we may assume that

$$(11) \quad \psi'' + \frac{s}{12}\psi = 0.$$

Since  $B$  is complete,  $\psi$  has to be defined on the whole of  $\mathbf{R}$ . Moreover, since  $B = S^1$ ,  $\psi$  has to be periodic. Therefore we may conclude that  $\psi$  is zero somewhere on  $B$ ; if  $\psi \neq 0$ ,

$$0 = \int_B \left( \frac{\psi'}{\psi} \right)' = \int_B \frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} = - \int_B \left( \frac{s}{12} + \frac{\psi'^2}{\psi^2} \right) < 0$$

which is a contradiction. Hence the given warped product metric  $g$  can not be a solution of (3).  $\square$

LEMMA 8. Let  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with the 2-dimensional complete base  $B$ . Suppose that  $f$  is a function of  $B$  only. Then  $g$  can not be a solution of (3).

A proof of Lemma 8 need another section and will be presented in the Section 4.

LEMMA 9. Let  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with the 2-dimensional complete base  $B$ . Then  $g$  can not be a solution of (3). In other words, Case 2 does not occur.

*Proof.* In the beginning, the proof of Lemma 9 may go along the same line as the proof of Lemma 7, concluding that

$$(12) \quad f = \psi b + c,$$

where  $b$  is a function on  $F$  and  $c$  is a function on  $B$ . By Proposition 4 and  $(1 + f) \sum_{i=1}^2 z(X_i, X_i) = \sum_{i=1}^2 \langle D_{X_i} df, X_i \rangle + \frac{sf}{6}$ , we have

$$(13) \quad (1 + f)(\check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}) = \sum_{i=1}^2 \langle D_{X_i} df, X_i \rangle + \frac{sf}{6}.$$

Now, substitution (12) into (13) gives

$$(14) \quad (1 + \psi b + c)(\check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}) = b \check{\Delta} \psi + \check{\Delta} c + \frac{s}{6}(\psi b + c),$$

where we used the fact that

$$\begin{aligned} \sum_{i=1}^2 \langle D_{X_i} df, X_i \rangle &= \sum_{i=1}^2 (\langle D_{X_i} \check{d}f, X_i \rangle + \langle D_{X_i} \hat{d}f, X_i \rangle) = \sum_{i=1}^2 \langle D_{X_i} \check{d}f, X_i \rangle \\ &= b \sum_{i=1}^2 \langle D_{X_i} d\psi, X_i \rangle + \sum_{i=1}^2 \langle D_{X_i} dc, X_i \rangle = b \check{\Delta} \psi + \check{\Delta} c \end{aligned}$$

with  $D_{X_i} \hat{d}f = \mathcal{V} D_{X_i} \check{d}f$ . It is easy to see that (14) can be rewritten as

$$(15) \quad b(\psi \check{s} - 3 \check{\Delta} \psi - \frac{2s}{3} \psi) = \check{\Delta} c - (c + 1)(\check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}) + \frac{s}{6} c.$$

Note that both  $\psi \check{s} - 3 \check{\Delta} \psi - \frac{2s}{3} \psi$  and the right-hand side are functions of  $B$  only, while  $b$  is a function of  $F$  only. Thus, in order that the equation (15) holds, either  $b$  is constant or  $\psi \check{s} - 3 \check{\Delta} \psi - \frac{2s}{3} \psi$  has to be zero. If  $b$  is constant,  $f$  is a function of  $B$  only, and so  $g$  can not be a solution of (3) in virtue of Lemma 8. Thus we may assume that

$$(16) \quad \check{s} = \frac{3 \check{\Delta} \psi}{\psi} + \frac{2}{3} s$$

and

$$(17) \quad \check{\Delta} c - (c + 1)(\check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}) + \frac{s}{6} c = 0.$$

Substitution of (16) into (17) gives

$$(18) \quad \psi \check{\Delta} c - (c+1) \check{\Delta} \psi - \frac{s}{6} \psi = 0.$$

Now, integrating both sides of (18) over  $B$ , we have

$$\frac{s}{6} \int_B \psi = 0,$$

where we used the fact that  $\int_B \psi \check{\Delta} c = \int_B c \check{\Delta} \psi$ . This equation together with the condition that  $\psi > 0$  leads to a contradiction.  $\square$

LEMMA 10. *Let  $(M, g) = (B, \check{g}) \times_{\psi^2} (F, \hat{g})$  with the 3-dimensional complete base  $B$ . Then  $g$  can not be a solution of (3). In other words, Case 3 does not occur.*

*Proof.* Suppose that the given warped product metric is a solution of CPE. From the definition of the Laplacian  $\Delta$ , we have

$$(19) \quad \langle D_U df, U \rangle = \Delta f - \check{\Delta} f = -\frac{s}{3} f - \check{\Delta} f$$

and from CPE we also have

$$(20) \quad (1+f) \left( -\frac{\check{\Delta} \psi}{\psi} - \frac{s}{4} \right) = \langle D_U df, U \rangle + \frac{s}{12} f.$$

Hence, the following relation holds by combining (19) and (20):

$$(1+f) \left( -\frac{\check{\Delta} \psi}{\psi} - \frac{s}{4} \right) = -\check{\Delta} f - \frac{s}{4} f$$

which is equivalent to

$$(21) \quad (1+f) \check{\Delta} \psi + \frac{s}{4} \psi = \psi \check{\Delta} f.$$

Now, integrating both sides of (21) over  $B$ , we have

$$\int_B f \check{\Delta} \psi + \frac{s}{4} \int_B \psi = \int_B \psi \check{\Delta} f = \int_B f \check{\Delta} \psi$$

which implies that

$$\int_B \psi = 0.$$

This equation together with the condition that  $\psi > 0$  leads to a contradiction.  $\square$

#### 4. Proof of Lemma 8

This section is devoted to the proof of Lemma 8. Suppose that  $g$  is a solution of (3), namely CPE, and  $f$  is a function of  $B$  only. In the following two contentions, we shall prove that this assumption leads to a contradiction. We first consider the following two cases according to the values of  $f$ . In the first case that  $f$  always takes values greater than or equal to  $-1$ , the proof of the Lemma is completed since  $(M, g)$  is isometric to  $S^4$  in virtue of Theorem 2. In the second case that  $f$  takes a value less than  $-1$ , this Lemma is proved in a series of two contentions under present conditions<sup>1</sup>. In this case, there exists a non-empty set  $H = \{x \in B \mid f(x) = -1\}$ . After investigating the analytic properties of the tensor  $D_X X$  for a tangent vector field  $X$  to  $H$  in Contention 1, we prove in Contention 2 that the present conditions give a contradiction. Since this contradiction is obtained from the assumption that  $g$  is a solution of CPE, we may conclude that  $g$  can not be a solution of CPE, proving our Lemma.

CONTENTION 1. Under present conditions, we have  $D_X X = -\frac{s}{12} df$  on  $H$ , where  $H = \{x \in B \mid f(x) = -1\}$ ,  $X \in TB$  is a tangent vector field to  $H$ , and  $N \in TB$  is a normal vector field to  $H$ .

*Proof.* We may assume that  $H$  is non-empty; otherwise  $g$  is Einstein by Theorem 2. Also, we note in [4] that a point of  $H$ , which is a critical point of  $f$ , is a non-degenerate local minimum point of  $f$ , and that such non-degenerate critical points are isolated. Therefore,  $H$  is a set consisting of finite critical points of  $f$ , or hypersurfaces of  $M$ , or union of both.

Putting  $W = |df|^2$ , it was proved in [4] that  $W$  is constant in each component of  $H$  and does not vanish on  $H$ . Therefore, in a small tubular neighborhood of  $H$ , we may take orthonormal frame fields  $\{X, N\}$ , where  $N = \frac{df}{W^{1/2}}$  on  $H$ . From (3), we have

$$(22) \quad Ddf = \frac{s}{12} g$$

on  $H$ . Then, it follows that on  $H$  we have

$$\frac{s}{12} = \langle D_X df, X \rangle = X \langle df, X \rangle - \langle df, D_X X \rangle = -\langle df, D_X X \rangle.$$

<sup>1</sup>In the proof of Contention 1 and 2, we assume that the metric  $g$  of warped product  $(M, g) = (B, \hat{g}) \times_{\psi^2} (F, \hat{g})$  is a solution of (3). We also suppose that  $\dim B = 2$  and  $f$  takes a value less than  $-1$ . Hereafter, this situation will be described by the words “present conditions”.

Hence,  $D_X X = -\frac{s}{12} \frac{1}{W^{\frac{1}{2}}} N$  on  $H$ .  $\square$

CONTENTION 2. Under present conditions, we have a contradiction on  $H$ .

*Proof.* In virtue of Proposition 4, (3) may be reduced to

$$(23) \quad (1+f)\left(\check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}\right) = \check{\Delta} f + \frac{sf}{6}$$

$$(24) \quad (1+f) \left( \frac{\hat{s}}{\psi^2} - 2 \frac{\check{\Delta} \psi}{\psi} - 2 \frac{|d\psi|^2}{\psi^2} - \frac{s}{2} \right) = 2 \frac{\langle d\psi, df \rangle}{\psi} + \frac{sf}{6}$$

since

$$(25) \quad \sum_{i=1}^2 z(X_i, X_i) = \check{s} - \frac{2}{\psi} \check{\Delta} \psi - \frac{s}{2}$$

$$(26) \quad \sum_{i=1}^2 z(U_i, U_i) = \frac{\hat{s}}{\psi^2} - 2 \frac{\check{\Delta} \psi}{\psi} - 2 \frac{|d\psi|^2}{\psi^2} - \frac{s}{2}$$

and  $\check{g}(d\psi, df) = \langle d\psi, df \rangle$ . Hence, using (23) and (24) we have

$$(27) \quad \check{\Delta} f = \frac{s}{6}$$

$$(28) \quad \frac{\langle d\psi, df \rangle}{\psi} = \frac{s}{12}$$

on  $H$ . The relation  $\sum_{i=1}^2 z(X_i, X_i) + \sum_{i=1}^2 z(U_i, U_i) = 0$  gives

$$(29) \quad (1+f)\left(-\check{s} + \frac{2}{\psi} \check{\Delta} \psi + \frac{s}{2}\right) = 2 \frac{\langle d\psi, df \rangle}{\psi} + \frac{sf}{6}.$$

Taking the Lie derivative of (29) with respect to  $df$  on  $H$ , we have

$$\begin{aligned} & W \left( -\check{s} + \frac{2}{\psi} \check{\Delta} \psi + \frac{s}{2} \right) \\ &= 2 \frac{\langle D_{df} d\psi, df \rangle}{\psi} + 2 \frac{\langle d\psi, D_{df} df \rangle}{\psi} - 2 \frac{\langle d\psi, df \rangle^2}{\psi^2} + \frac{s}{6} W \\ &= 2 \frac{\langle D_N d\psi, N \rangle}{\psi} W + \frac{s}{6} W, \end{aligned}$$

where

$$2 \frac{\langle d\psi, D_{df} df \rangle}{\psi} = \frac{s}{6\psi} \langle df, d\psi \rangle = \frac{s^2}{72} = 2 \frac{\langle d\psi, df \rangle^2}{\psi^2}$$

in virtue of (22) and (28). Therefore

$$(30) \quad -\check{s} + \frac{2}{\psi} \langle D_X d\psi, X \rangle + \frac{s}{3} = 0,$$

where we used the fact that

$$(31) \quad \check{\Delta}\psi = \langle D_X d\psi, X \rangle + \langle D_N d\psi, N \rangle.$$

On the other hand, in order to calculate  $z(X, X)$  we take the Lie derivative of (3) with respect to  $df$  on  $H$ . Then

$$\begin{aligned} Wz(X, X) &= \langle D_{df} D_X df, X \rangle + \langle D_X df, D_{df} X \rangle + \frac{s}{12} W \\ &= W^{\frac{1}{2}} \langle D_N D_X df, X \rangle + |D_X df|^2 + \frac{s}{12} W \\ (32) \quad &= W^{\frac{1}{2}} \langle D_N D_X df, X \rangle + \frac{s^2}{144} + \frac{s}{12} W \end{aligned}$$

on  $H$ , where we used the fact that  $W$  is constant on  $H$ ,  $D_{df} X = W^{-\frac{1}{2}} D_N X = W^{-\frac{1}{2}} D_X N = D_X df$ , and  $D_X df = \frac{s}{12} X$  on  $H$ . However, the relation  $D_N D_X df = D_X D_N df + R(X, N)df$  gives

$$\begin{aligned} \langle D_N D_X df, X \rangle &= \langle D_X D_N df, X \rangle + \langle R(X, N)df, X \rangle \\ &= X \langle D_N df, X \rangle - \langle D_N df, D_X X \rangle - W^{\frac{1}{2}} K(X, N) \\ &= \frac{s}{12W^{\frac{1}{2}}} \langle D_N df, N \rangle - W^{\frac{1}{2}} \check{r}(N, N) \\ (33) \quad &= \frac{s^2}{144W^{\frac{1}{2}}} - W^{\frac{1}{2}} \check{r}(N, N), \end{aligned}$$

where we used the fact that  $\langle D_N df, X \rangle = 0$ ,  $D_X X = -\frac{s}{12} \frac{1}{W^{1/2}} N$  on  $H$ , (22), and  $\check{r}(N, N) = K(X, N)$ . Now substituting (33) into (32), we have

$$(34) \quad z(X, X) = \frac{s^2}{72W} - \check{r}(N, N) + \frac{s}{12}.$$

Note that from Proposition 4 we have

$$(35) \quad z(X, X) = r(X, X) - \frac{s}{4} = \check{r}(X, X) - 2 \frac{\langle D_X d\psi, X \rangle}{\psi} - \frac{s}{4}.$$

Substituting (34) into (35) on  $H$  we have

$$(36) \quad \check{s} - 2 \frac{\langle D_X d\psi, X \rangle}{\psi} - \frac{s}{3} - \frac{s^2}{72W} = 0.$$

The equation (36) clearly contradicts the equation (30). This contradiction comes from the assumption that  $g$  is a solution of CPE. Hence,  $g$  can not be a solution of CPE, proving our Lemma 8.  $\square$

## 5. A rigidity result

This section is devoted to the proof of the following rigidity result.

**PROPOSITION 11.** *Let  $M$  be a compact oriented 4-dimensional manifold containing an open dense subset  $U$  which is (for the induced metric) a warped product on a 1-dimensional basis with Einstein fibre  $F$ . If  $g$  is a solution metric of (3), then  $(M, g)$  is isometric to the standard sphere  $S^4$ .*

We take  $U$  as large as possible. If the base  $B$  is complete, then  $M$  is a warped product and Lemma 7 applies. If not,  $B$  is an open interval  $(0, a)$  with  $U = (0, a) \times F$ . It is easy to see that  $M$  is the quotient of  $[0, a] \times F$ . Note that the following relation holds in order for  $M$  to be a complete manifold:

$$(37) \quad \psi(0) = \psi(a) = 0.$$

Note also that the following relation holds in order for  $M$  to be smooth (c.f. [7]):

$$(38) \quad \psi'(0) = -\psi'(a) = \sqrt{\frac{\hat{s}}{6}}.$$

Thus (37) and (38) become the initial conditions for the warping function  $\psi$ . On the other hand, in virtue of Proposition 4,  $\psi$  satisfies

$$(39) \quad 6\psi''\psi + 6\psi'^2 + \psi^2 s - \hat{s} = 0$$

with constants  $s$  and  $\hat{s}$ . For a solution  $\psi$  of (39), we observe that  $\psi''(0) = 0$ , since the differentiation of (39) gives  $3\psi''\psi + 9\psi''\psi' + s\psi\psi' = 0$ . It is easy to see that  $\psi_0 = \sqrt{\frac{2\hat{s}}{s}} \sin \sqrt{\frac{s}{12}}t$  with  $a = \pi\sqrt{\frac{s}{12}}$  is a solution of the ordinary differential equation (39) with the initial conditions (37) and (38). The following lemma shows that the solution of (39) is unique.

**LEMMA 12.**  *$\psi_0$  is the unique solution of (39) under the conditions (37) and (38). Therefore,  $(M, g) = [0, a] \times_{\psi_0} S^3$ .*

*Proof.* Let  $\psi$  be another solution of (39) with the initial conditions (37) and (38), and let  $F = \frac{\psi}{\psi_0}$ . It is easy to see that  $F$  is well-defined on  $[0, a]$  with  $F(0) = 1$ , since  $\psi_0(0) = \psi(0) = 0$  and  $\psi'_0(0) = \psi'(0) = \sqrt{\frac{\hat{s}}{6}}$ . Now, in order to prove our Lemma, we claim that  $F \equiv 1$ , implying that  $\psi = \psi_0$ . First we observe that  $F'(0) = 0$  since  $\psi'' = \psi''_0 F + 2\psi'_0 F' + \psi_0 F''$

and  $\psi_0''(a) = \psi''(0) = 0$ . Now, if we substitute  $\psi = \psi_0 F$  into (39),  $F$  satisfies

$$(40) \quad F'' = -\frac{1}{\psi_0^2 F} \left( 4\psi_0 \psi_0' F F' + \psi_0^2 F'^2 + \frac{1}{6} \hat{s}(F^2 - 1) \right),$$

where we used the fact that  $\psi_0$  satisfies (39). Define a function  $\xi : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  by

$$\xi(t, y_1, y_2) = -\frac{1}{\psi_0^2 y_1} \left( 4\psi_0 \psi_0' y_1 y_2 + \psi_0^2 y_1'^2 + \frac{1}{6} \hat{s}(y_1^2 - 1) \right),$$

where  $\xi$  is obtained simply by replacing  $F$  and  $F'$  in (40) by  $y_1$  and  $y_2$  respectively. In order to prove the uniqueness of solutions of (40) satisfying  $F(0) = 1$  and  $F'(0) = 0$ , it suffices to show that  $\xi(t, y_1, y_2)$  is continuous with respect to  $t$  and is Lipschitz with respect to  $y_1$  and  $y_2$  in the rectangle  $R = [0, a] \times [r, R_1] \times [-R_2, R_2]$  for all  $0 < r < 1 < R_1$  and  $R_2 > 0$ . First, it is easy to see that  $\xi$  is continuous for  $t$  by letting  $\xi(0, y_1(0), y_2(0)) = 0$ . Secondly,  $\xi$  is Lipschitz, since  $\xi$  is smooth with respect to  $y_1, y_2$  in  $R$ , and  $|\xi(t, y_1(t), y_2(t)) - \xi(t, \bar{y}_1(t), \bar{y}_2(t))| < M(|y_1(t) - \bar{y}_1(t)| + |y_2(t) - \bar{y}_2(t)|)$  for some  $M > 0$ . Therefore, the ordinary differential equation (40) has the unique solution  $F \equiv 1$ , proving our claim.  $\square$

It is well known (see [1], Corollary 9.107) that the warped product  $M = B \times_{\psi^2} F$  is Einstein if and only if  $\hat{g}, \hat{g}, \psi$  satisfy that  $(F, \hat{g})$  is Einstein, and

$$(41) \quad -\frac{\psi''}{\psi} - 3\frac{\psi'^2}{\psi^2} + \frac{\hat{s}}{3\psi^2} = \frac{s}{4}$$

and

$$(42) \quad -3\frac{\psi''}{\psi} = \frac{s}{4}.$$

It is an easy exercise for  $\psi = \psi_0$  to satisfies (41), (42). Also note that  $F$  is Einstein from the assumption. Therefore we may conclude that the warped product  $U$  (or  $M$ ) is Einstein. However, in virtue of Theorem 3,  $M$  must be isometric to a standard sphere.

ACKNOWLEDGEMENT. The authors would like to express their deep thanks to the referee for valuable suggestions to improve the paper.

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