

Intelligent computing analysis of radiative heat transfer in magnetized bio-convective Williamson fluid flow over a curved elongated surface

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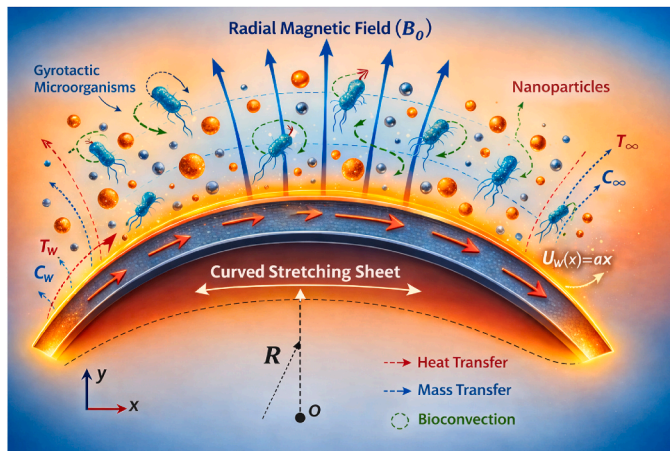
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GRAPHICAL ABSTRACT



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ABSTRACT

Considering diverse applications of bioconvective nanofluid in bio-medical and other relevant of science and technology, strive is made to explore Williamson nanofluid using bioconvection for

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 Artificial neural network

the swimming motion gyrotactic microorganisms in a stretched curved surface. A mathematical model is presented, yielding PDES, which are subjected to similarity transformations for simplification. The solution for the radiative induced nanofluid is explored with BVP5C solution which is validated using Artificial Neural Network (ANN) based Levenberg–Marquardt back-propagation. Outstanding unanimity of BVP5C solution is revealed with ANN, showing precision and accuracy of the solution scheme. The present study examines rheological nanofluid transport over a curved stretching sheet, incorporating key effects such as thermophoresis, Brownian motion, velocity and concentration slip, and motile microorganism density. The nonlinear system is solved numerically using by employing robust BVP5C method and validated through an ANN based on the Levenberg–Marquardt algorithm. The influence of governing parameters on velocity, temperature, concentration, skin friction, Sherwood number, and microorganism density is analyzed. The outcomes of investigation indicate that curvature and magnetic effects hinder fluid motion, while increased curvature enhances gyrotactic microorganism density. Higher radiation raises fluid temperature and thickens the thermal boundary layer, whereas thermal slip reduces heat transfer near the wall. Increasing the Prandtl number suppresses thermal diffusion, leading to a thinner thermal boundary layer. The strong agreement between numerical and ANN results confirms the model's accuracy. These findings of the reported fluidic scheme cover diverse applications in biomedical engineering, microfluidics, and biotechnology, including drug delivery and bioconvective flow control.

1. Introduction

Nanofluids are advanced heat transfer fluids prepared by adding nanoparticles to base fluids provide trigger to heat transfer of liquids. Nanofluids have extensive applications in various industries like in the working of electronic devices, cancer hyperthermia, biochemical engineering, magnetic resonance imaging (MRI), lubricants, environmental cleanup, catalysis, and various geothermal power stations. Inspired by huge significance and their application many researchers are doing lots of research to explore this avenue. Gadelhak et al. [1] transport of Williamson nanofluid model through a curved stretching sheet and explore the significance in thermal engineering processes, by using numerical technique with Buongiorno model. The transport of bioconvective nanofluid with heat and mass transfer were investigated [2] for gyrotactic microorganisms along with magnetic field through a curved stretching surface. The non-Newtonian time dependent Williamson fluid with thermophoresis and Brownian effects were reported [3,4] with time variant electrically conducting magnetic field explored the solution with numerical techniques. The effects of magnetic forces combined with nonlinear thermal radiation, using gold and copper nanoparticles in a peristaltic channel, were investigated in Refs. [5,6] for both high and low Reynolds numbers. The momentum, Maxwell, and energy equations were solved by incorporating nonlinear thermal radiation and heat source effects.

The transport of 3D Williamson nanofluid over a stretching sheet with the Cattaneo–Christov heat and mass fluxes were reported [7,8] their investigation incorporates thermophoresis, thermal radiation, and Brownian motion and showed that non-Newtonian nature of the fluid significantly velocity and temperature profiles. Flow dynamics for 2D Williamson fluid model by incorporating the ternary hybrid nanofluids with gyrotactic microorganisms with heat transfer characteristics investigated [9–11], their study comprises of viscous dissipation formulations by employing Rosseland approximation. The swimming motion of microorganisms were investigated [12–15] for parasitically induced flow with comprehensive deliberations using Rabinowitsch fluid by incorporating the buoyancy effect with Boussinesq approximation.

Magnetohydrodynamics (MHD) is very crucial phenomenon, and it is the study of the motion of electrically conducting fluids like liquid metals, plasmas, saltwater, or ionized nanofluids, by employing the influence of magnetic fields. It has been observed that when magnetic field interact with a moving conductive fluid, it generates a Lorentz force which significantly influences the flow by boosting its flow or slows down, or redirect the flow, depending on the field strength and its directions. Merger of electromagnetic forces and fluid motion makes MHD vital for controlling heat and mass transfer in complex engineering systems. MHD effects are widely applied in fusion reactors, electromagnetic pumps, cooling systems in nuclear engineering, metallurgical processing, and astrophysical plasma modeling. In biomedical flows, MHD is used to regulate drug transport in magneto nanofluids and enhance thermal therapies, and it provides a powerful mechanism for manipulating conductive fluids in both industrial and scientific applications. Sheikholeslami and Ellahi [16] studied the MHD effects on 3D nanofluid transport within three-dimensional coordinates and reported that MHD forces enhance the drag force while reducing convective currents and the rate of thermal transfer. The thermophysical axioms of carbon nanotubes were explored in Refs. [17–19] for MHD-influenced fluid transport over a stretching surface. Borrelli et al. [20] examined oblique stagnation-point flow under the influence of MHD for a steady-state case and further reported that as the electric field weakens, the magnetic field reorients itself within the flow plane. Ahmed et al. [21] presented a comprehensive study of MHD Williamson nanofluid flow through a porous, exponentially stretching curved surface, considering the effects of activation energy and variable thermal conductivity. Considering the heat transfer for cilia-induce MHD transport of rheological fluid in a curved microchannel investigated [22–25], these investigations were carried out by employing heat source/sink, Joule heating, thermal radiation, mixed convection and viscous dissipation. Several significant investigations on MHD non-Newtonian nanofluids over stretching surfaces have been reported recently [26–29]. Heat and mass transfer analysis were reported [30–32] by focusing bioconvective transport of rheological hybrid nanofluid, they integrated nonlinear thermal radiation and chemical reaction effects. Bioconvection having

biotechnological applications occurs due to the collective motion of self-propelled microorganisms, generating complex flow patterns that enhance mixing.

Due to their enhanced thermal conductivity, nanofluids play a vital role in modern heat transfer and cooling technologies, including electronics cooling, heat exchangers, and solar collectors. In this context, the behavior of partially ionized fluids were investigated [33–35] over a convectively heated cylinder under the influence of a magnetic field. The Darcy-Forchheimer bio-convective transport of a Second-grade nanofluid were explored [36,37] by considering applications in chemical processes, bioengineering and food industry over a porous stretching cylinder.

After a comprehensive literature review, it is revealed that the simultaneous incorporation of radiative heat transfer, curvature effects, velocity slip, and bioconvection of gyrotactic microorganisms within an MHD Williamson nanofluid model has rarely been addressed collectively in existing studies. The governing PDEs are transformed into similarity-based ODEs and solved using a hybrid approach that combines BVP5C with ANN-based Levenberg–Marquardt backpropagation for enhanced validation. Reported investigation provides new insights into microorganism dynamics under the coupled influence of magnetic field and curvature. Additionally, the evaluation of transport characteristics such as skin friction and Sherwood number under these combined effects highlights its practical significance.

Following are the objective current investigations.

- To formulate and simplify a radiative nanofluid model by converting the governing PDEs into ODEs using similarity transformations.
- To obtain and validate numerical solutions of the transformed system using BVP5C and verify their accuracy through ANN-based Levenberg–Marquardt backpropagation.
- To analyze the physical effects of key transport mechanisms such as Brownian motion, thermophoresis, curvature, magnetic field, and slip conditions on velocity, temperature, and concentration fields.
- To investigate the behavior of bioconvection and microorganism dynamics, particularly the influence of curvature and magnetic effects on gyrotactic microorganism density.
- To evaluate skin friction, Sherwood number, and thermal/mass transport characteristics to support applications in biomedical engineering, microfluidics, and bioprocessing technologies.

2. Formulation of the problem

A steady incompressible transport of MHD Williamson nanofluid with heat, mass transfer and with gyrotactic microorganisms is focused in a linear curved stretching sheet. The mathematical model is formulated by assuming a steady, incompressible, and laminar Williamson nanofluid flow, incorporating shear-thinning behavior with negligible infinite shear viscosity and a small Weissenberg number, which are well supported by rheological experiments [38–40]. The effects of a uniform magnetic field, thermal radiation (via utilizing Rosseland approximation), and nanoparticle transport through Brownian motion and thermophoresis are included based on experimentally validated heat and mass transfer mechanisms. In addition, gyrotactic microorganisms, velocity slip at the boundary, and negligible induced magnetic field are considered, consistent with observations from microfluidic and bioconvection. The analyze the flow scheme $(\tilde{r}, \tilde{x})$ coordinates system is used in such manner that $\tilde{x}$ is along with flow, and $\tilde{r}$ is transverse to it. $\tilde{R}$ is radius from

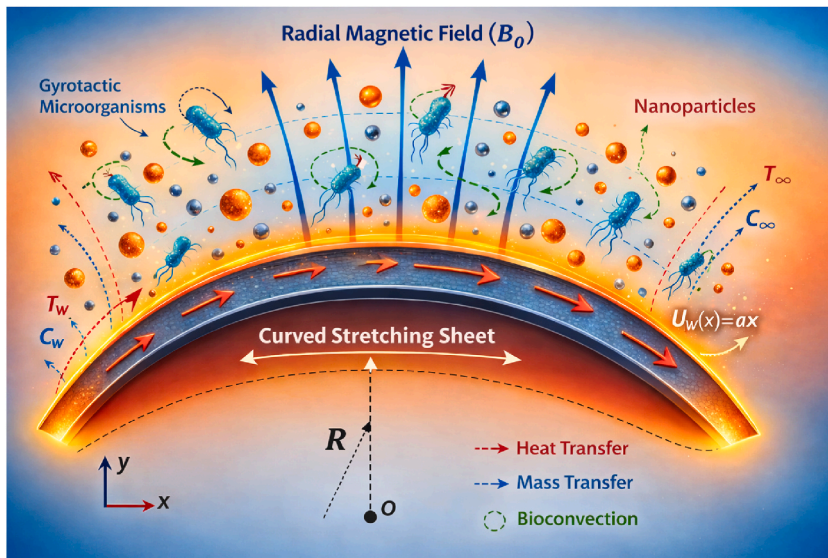


Fig. 1. Exhibits geometry of the curved stretching sheet filled with nanoparticles and microbes.

origin along curved, flow scheme is exhibited in Fig. 1. The velocity of curved stretching sheet is $V_w = -a_1 x$, curved surface is stretched along axial direction. The magnetic field $\mathbf{B} = \left(\frac{B_0 \tilde{R}}{\tilde{R} + \tilde{r}}, 0, 0 \right)$ is applied in radial direction [2,26].

Stress tensor of Williamson fluid [38–40] model is

$$\check{\tau} = \check{\mu}_\infty + \frac{\check{\mu}_0 - \check{\mu}_\infty \check{A}_1}{1 - \check{\Gamma} \check{\gamma}} \quad (1)$$

$\check{A}_1$ is the first Rivlin, Erickson tensor, $\check{\mu}_0, \check{\mu}_\infty$ representing zero and finite shear viscosities, $\check{\Gamma}$ characteristics time, and $\check{\gamma}$ representing shear rate having expression $\check{\gamma} = \sqrt{\frac{1}{2} \check{D}}$, $\check{D} = tr(\check{A}_1^2)$. In this investigation $\check{\mu}_\infty = 0$ is considered, and $\check{\Gamma} \check{\gamma} \ll 1$. By taking these considerations the stress tensor may be presented as $\check{\tau} = \frac{\check{\mu}_0}{1 - \check{\Gamma} \check{\gamma}} \check{A}_1$ and further applying the binomial expansion the stress tensor takes the shape $\check{\tau} = \check{\mu}_0 [1 - \check{\Gamma} \check{\gamma}] \check{A}_1$.

By implementing the boundary layer approximations, equations of motion for the Williamson fluid for Buongiorno's nanofluid mode [2,26–30,41,42].

$$\check{R} \frac{\partial \check{v}}{\partial \check{x}} + \frac{\partial}{\partial \check{r}} (\check{u}(\check{r} + \check{R})) = 0, \quad (2)$$

$$\rho_f \left[-\frac{\check{v}^2}{\check{R} + \check{r}} \right] = -\frac{\partial \check{p}}{\partial \check{r}}, \quad (3)$$

$$\begin{aligned} \left[\check{u} \frac{\partial \check{v}}{\partial \check{r}} + \frac{\check{v} \check{R}}{\check{R} + \check{r}} \left(\frac{\partial \check{v}}{\partial \check{x}} \right) + \frac{\check{v} \check{u}}{\check{R} + \check{r}} \right] = & - \left(\frac{\check{R}}{\check{R} + \check{r}} \right) \frac{1}{\rho} \left(\frac{\partial \check{p}}{\partial \check{x}} \right) + \check{v}_s \frac{\partial^2 \check{v}}{\partial \check{r}^2} + \check{v} \frac{\check{\Gamma}}{\check{R} + \check{r}} - \left(\frac{\partial \check{v}}{\partial \check{r}} \right) \left(\frac{2\check{v}}{\check{R} + \check{r}} \right) + \left(\frac{\partial \check{v}}{\partial \check{r}} \right)^2 + \left(\frac{\check{v}}{\check{R} + \check{r}} \right)^2 \\ & + 2\check{v} \check{\Gamma} \left(\frac{\partial \check{v}}{\partial \check{r}} \frac{\partial^2 \check{v}}{\partial \check{r}^2} - \left(\frac{\partial^2 \check{v}}{\partial \check{r}^2} \right) \left(\frac{\check{v}}{\check{R} + \check{r}} \right) + \left(\frac{2\check{v}}{(\check{R} + \check{r})^2} \right) \left(\frac{\partial \check{v}}{\partial \check{r}} \right) - \left(\frac{\partial \check{v}}{\partial \check{r}} \right)^2 \left(\frac{1}{\check{R} + \check{r}} \right) - \left(\frac{\check{v}^2}{(\check{R} + \check{r})^3} \right) \\ & - \left(\frac{\check{R} B_0}{\check{R} + \check{r}} \right)^2 \check{v} \sigma_f \end{aligned} \quad (4)$$

$$\left[\check{u} \frac{\partial \check{T}}{\partial \check{r}} + \frac{\check{R} \check{v}}{\check{R} + \check{r}} \frac{\partial \check{T}}{\partial \check{x}} \right] = \frac{K_f}{(\rho C_p) f} \left(1 + \frac{16\sigma^* T_\infty^3}{3 K_f k^*} \right) \left[\frac{1}{\check{R} + \check{r}} \frac{\partial \check{T}}{\partial \check{r}} + \frac{\partial^2 \check{T}}{\partial \check{r}^2} \right] + \tau_* \left[D_B \frac{\partial \check{T}}{\partial \check{r}} \frac{\partial \check{C}}{\partial \check{r}} + \frac{D_T}{T_\infty} \left(\frac{\partial \check{T}}{\partial \check{r}} \right)^2 \right] \quad (5)$$

$$\left[u \frac{\partial \check{C}}{\partial \check{r}} + \frac{\partial \check{C}}{\partial \check{x}} \left(\frac{\check{R} \check{v}}{\check{R} + \check{r}} \right) \right] = D_B \left[\frac{1}{\check{R} + \check{r}} \frac{\partial \check{C}}{\partial \check{r}} + \frac{\partial^2 \check{C}}{\partial \check{r}^2} \right] + \frac{D_T}{T_\infty} \left[\left(\frac{\partial \check{T}}{\partial \check{r}} \right) \left(\frac{1}{\check{R} + \check{r}} \right) + \frac{\partial^2 \check{T}}{\partial \check{r}^2} \right] \quad (6)$$

$$\left[u \frac{\partial \check{N}}{\partial \check{r}} + \frac{\partial \check{N}}{\partial \check{x}} \left(\frac{\check{R} \check{v}}{\check{R} + \check{r}} \right) \right] = D_m \left[\frac{1}{\check{r} + \check{R}} \frac{\partial \check{N}}{\partial \check{r}} + \frac{\partial^2 \check{N}}{\partial \check{r}^2} \right] - b \frac{W_c}{C_w - C_\infty} \left[\frac{\partial}{\partial \check{r}} \left(\check{N} \frac{\partial \check{C}}{\partial \check{r}} \right) \right] \quad (7)$$

In the above equations $\check{v}_s = \left(\frac{\mu_f}{\rho_f} \right)$ is kinematic viscosity, $\tau_* = \left(\frac{\rho_c p}{\rho_c p} \right)_p$ ratio of heat capacitance, p is the pressure, W_c the optimum cell speed, D_B Brownian motion parameter, T_∞ representing ambient temperature, K_f signifies thermal conductivity, D_T is thermophoresis, B_0 is the constant magnetic field, k^* is spectral absorption coefficient, D_m , representing concentration of motile organisms, σ_f is the electrical conductivity of fluid, σ^* is Stefan Boltzmann constant, C_w wall concentration, and $\check{N}$ density of gyrotactic microorganisms. Fig. 1 represents a curved stretching sheet which is a deformable surface with curvature that stretches along its length, it significantly influences fluid transport and boundary layer development compared to flat geometries. This nature is particularly important in processes like coating, extrusion, and biomedical flows. The geometry of the curved stretching surface [25,29,30] is represented through B. Cs that incorporate linear stretching along the surface and curvature effects.

The adjoining B. Cs of the physical flow scheme are

$$\begin{aligned} \check{v} = u_w + L_1 \frac{\partial \check{v}}{\partial \check{r}} = a \check{x} + L_1 \frac{\partial \check{v}}{\partial \check{r}}, \check{u} = 0, \check{T} = T_w + L_2 \frac{\partial \check{T}}{\partial \check{r}} = T_0 + L_2 \frac{\partial \check{T}}{\partial \check{r}} + b_2 \check{x}, \\ \check{C} = C_w + L_3 \frac{\partial \check{C}}{\partial \check{r}} = C_0 + L_3 \frac{\partial \check{C}}{\partial \check{r}} + b_3 \check{x}, \check{N} = N_w + L_4 \frac{\partial \check{N}}{\partial \check{r}} = N_0 + L_4 \frac{\partial \check{N}}{\partial \check{r}} + b_4 \check{x} \text{ at } \check{r} = 0, \\ \check{v} \rightarrow 0, \check{T} = T_\infty, \check{C} = C_\infty, \check{N} = N_\infty \text{ as } \check{r} \rightarrow \infty. \end{aligned} \quad (8)$$

where L_1, L_2, L_3, L_4 representing velocity, thermal, concentration, and microorganisms' density slip parameters respectively. The velocity B. Cs includes a slip term, permitting partial fluid motion at the wall, which is more realistic for micro-scale flows.

Now implementing the similarity transformations as

$$\begin{aligned} \check{\nu} &= \alpha \check{x} f'(\eta), p = \rho \alpha^2 \check{x}^2 p(\eta), \check{u} = -\frac{\check{R}}{\check{R} + r} \sqrt{\alpha \check{\nu}} f(\eta), \Theta = \frac{\check{T} - \check{T}_\infty}{\check{T}_w - \check{T}_\infty}, \\ N &= \frac{\check{N} - \check{N}_\infty}{\check{N}_w - \check{N}_\infty}, \eta = \sqrt{\frac{\alpha}{\nu^*}} r, \Phi = \Theta = \frac{\check{C} - \check{C}_\infty}{\check{C}_w - \check{C}_\infty}, \end{aligned} \quad (9)$$

Now the above narrated similarity transformation into Eqs. (2)–(7), it is seen that Eq. (2) is identically satisfied and rest of the equations take the underneath shape:

$$p'(\eta) = \frac{1}{K + \eta} f'^2(\eta), \quad (10)$$

$$\begin{aligned} \frac{2K}{\eta + K} p(\eta) &= f''(\eta) + \frac{K}{\eta + K} f''(\eta) f(\eta) + \frac{K}{(K + \eta)^2} f'(\eta) f(\eta) - \frac{K}{\eta + K} f'^2(\eta) - \left(\frac{K}{\eta + K} \right)^2 \text{Hr} f'(\eta) \\ &+ W_e \left\{ \frac{1}{K + \eta} f'^2(\eta) - \frac{1}{K + \eta} f'^2(\eta) + \frac{2}{(\eta + K)^2} f''(\eta) f'(\eta) + 2 f''(\eta) f'(\eta) - 2 \frac{f''(\eta)}{\eta + K} f'(\eta) \right\} \end{aligned} \quad (11)$$

$$(1 + N_r) \left[\Theta'(\eta) + \frac{1}{\eta + K} \Theta'(\eta) \right] + Pr \frac{K}{\eta + K} f(\eta) \Theta'(\eta) + Pr N_b \Phi'(\eta) \Theta'(\eta) + Pr N_t (\Theta'(\eta))^2 = 0, \quad (12)$$

$$\frac{K}{\eta + K} S_c N_b f(\eta) \Phi'(\eta) + N_t \left[\Theta''(\eta) + \frac{1}{\eta + K} \Theta'(\eta) \right] + Nb \left[\phi''(\eta) + \frac{1}{\eta + K} \Phi'(\eta) \right] = 0, \quad (13)$$

$$N''(\eta) + \frac{1}{\eta + K} \chi'(\eta) + \frac{K}{\eta + K} S_b f(\eta) N'(\eta) - Pe [\Phi'(\eta) N'(\eta) - (N(\eta) + \tau_0) \Phi''(\eta)] = 0 \quad (14)$$

where $K = \sqrt{\frac{\alpha}{\nu^*}} \check{R}$ is the representative of curvature parameter, $\alpha = \alpha^*$ Williamson fluid parameter, $Re = \frac{\alpha \check{\nu}^2}{\nu^*}$ Reynolds number, $N_r = \frac{16\sigma^* \check{T}_\infty^3}{3k^* K_f}$ thermal radiation parameter, $Pr = \frac{(\rho C_p) \nu^*}{K_f}$ Prandtl number, $S_c = \frac{\nu^*}{D_B}$ Schmidt number, $N_t = \frac{(\check{C}_w - \check{C}_\infty) D_B \tau}{\nu^*}$ thermophoresis parameter, $\tau_0 = \frac{\check{N}_\infty}{\check{N}_w - \check{N}_\infty}$ is constant parameter, $N_b = \frac{(\check{T}_w - \check{T}_\infty) D_T \tau}{\nu^*}$ Brownian motion parameter, $Pe = \frac{W_e b}{D_m}$ is Peclet number, $S_b = \frac{\nu^*}{D_B}$, convective Schmidt number, $W_e = \alpha(Re)^{\frac{1}{2}}$ Weissenberg number and $H_r = \frac{\sigma_f B_0^2}{\rho_f a}$ Hartmann number.

Using Eq. (10) in Eq. (11)

$$\begin{aligned} \frac{4W_e f'[\eta]^2}{K + \eta} &+ \left(K + \eta \right) \left(-2(K - 2W_e) f''[\eta]^2 + 2(W_e(K + \eta) + K(1 + K + \eta) f[\eta]) f^{(3)}[\eta] + 4W_e(K + \eta)^2 f^{(3)}[\eta]^2 + 2(K + \eta)^2 f^{(4)}[\eta] \right. \\ &\left. + (K + \eta) f''[\eta] (KH_r - 4W_e f^{(3)}[\eta] + 4W_e(K + \eta) f^{(4)}[\eta]) \right) \\ &] = f'[\eta] (2Kf[\eta] + 2(K^3 + 4W_e + 2K^2\eta + K\eta^2) f''[\eta] + W_e(K + \eta) (KH_r - 4f^{(3)}[\eta] + 4(K + \eta) f^{(4)}[\eta])) \end{aligned} \quad (15)$$

The dimensionless average B. Cs are

$$\begin{aligned} f(0) &= 0, f'(0) - \beta_1 f''(0) = 1, \Theta(0) - \beta_2 \Theta'(0) = 1, \\ \Phi(0) - \beta_3 \Phi'(0) &= 1, N(0) - \beta_4 N'(0) = 1 \text{ at } \eta \rightarrow 0 \\ f' \rightarrow 0, \Theta \rightarrow 0, \Phi \rightarrow 0, N \rightarrow 0 &\text{ at } \eta \rightarrow \infty \end{aligned} \quad (16)$$

Physical quantities great significance like Skin friction $\frac{(Re)^{\frac{1}{2}} C_f}{2}$, Nusselt number $\frac{Nu}{(Re)^2}$, Sherwood number $\frac{Sh}{(Re)^2}$, gyrotactic microorganism number $\frac{M_h}{(Re)^{\frac{1}{2}}}$ [30–32] in dimensionless form are

$$\frac{(Re)^{\frac{1}{2}} C_f}{2} = \left[f''(0) - \frac{f'(0)}{K} - 2 \left(\alpha \frac{f'(0)}{K} - \alpha f''(0) \right) \sqrt{\left(\left(\frac{f(0)}{K} f'(0) \right)^2 + \frac{Re}{4} \left(-\frac{f'(0)}{K} + f''(0) \right)^2 \right)} \right] \quad (17)$$

$$\frac{Nu}{(Re)^{\frac{1}{2}}} = - (1 + N_r) \Theta'(0) \quad (18)$$

$$\frac{Sh}{(Re)^{\frac{1}{2}}} = - \Phi'(0) \quad (19)$$

$$\frac{M_n}{(Re)^{\frac{1}{2}}} = -N'(0) \quad (20)$$

2.1. BVP5C solution strategy

BVP5C is a MATLAB solver which is designed for solving boundary value problems (BVPs) using a fourth-order collocation method with adaptive mesh refinement. It tackles excellently to solve systems of ordinary ODEs subject to boundary conditions at two or more points. It automatically adjusts the mesh to achieve the desired accuracy while maintaining stability. It is outstandingly suitable for nonlinear, stiff, and coupled differential equation systems commonly found in fluid dynamics and heat transfer.

In this methodology all the ODES are converted into first order as:

$$\begin{aligned} f &= y(1), f' = y(2), f'' = y(3), f''' = y(4), \Theta = y(5), \Theta' = y(6), \Phi = y(7), \Phi' = y(8), N = y(9), N' = y(10) \\ dydx(4) &= (-KH_r W_e(K+x)y(2) + 2Ky(1)y(2) - (4W_e y(2)^2)/(K+x) + KH_r(K+x)^2 y(3) + 2(K^3 + 4W_e + 2K^2 x \\ &+ Kx^2)y(2)y(3) + 2(K - 2W_e)(K+x)y(3)^2 - 2(K+x)(W_e(K+x) + K(1+K+x)y(1))y(4) - 4W_e(K+x)y(2)y(4) \\ &+ 4W_e(K+x)^2 y(3)y(4) - 4W_e(K+x)^3 y(4)^2) / (2(K+x)^3 - 4W_e(K+x)^2 y(2) + 4W_e(K+x)^3 y(3)), \\ dydx(6) &= \frac{-N_r P r y(6)^2 - \frac{y(6)}{(K+x)}(1 + N_r + K P r y(1) + N b P r(K+x)y(8))}{1 + N_r}, \\ dydx(8) &= (-N_r y(6) - N_b y(8) - K N_b S_c y(1)y(8) - K N_r dydx(6) - N_r x dydx(6)) / (N_b(K+x)), \\ dydx(10) &= -(((1 + K S_b y(1) - P e(K+x)y(8))y(10)) / (K+x)) + P e(\tau_0 + y(9)) dydx(8) \end{aligned} \quad (21)$$

3. Results and discussion

A mathematical model is investigated, yielding PDES, which are subjected to similarity transformations for simplification. The solution for the radiative induced nanofluid is explored with BVP5C solution which is validated using Artificial Neural Network (ANN) based Levenberg Marquardt backpropagation. In this section the solution methodology is comprehensively studied with aid of ANN, graphical and tabular deliberations.

3.1. ANN solution

ANNs are brain inspired computational models [43,44] which are capable of learning from data. It efficiently capture complex and nonlinear relationships that traditional methods struggle hard to do so. In recent times ANNs due to their immense significance are widely used for prediction, classification, optimization, and function approximation. In the domain of scientific computing, ANN offer smooth, accurate solutions for complex BVPs systems. Their flexibility, adaptability, and strong generalization proved ANNs highly significant in modern engineering and research. The training dataset for the ANN is created from the numerical solution obtained from the outcomes of the BVP5C. In the employed solution scheme, the input data are the similarity variable values η ($x_solution$), while the target outputs are taken from the computed solution $f'(\eta)/\Theta(\eta)$ ($y_solution(2,:)$ and $_solution(5,:)$). These input/output pairs are directly used to train the NN. No explicit normalization or scaling technique is employed to the dataset in the current functioning. Regarding grid independence, a dense initial mesh of 1500 points is used, and the BVP5C solver automatically refines the mesh and controls the error, ensuring convergence and solution accuracy. Table No 1 provides key outcomes of BVP5C computation results with ANN predictions for the derivative $f'(\eta)$ for varying velocity slip parameter β_1 . It has been observed that training, validation, and test errors are extremely small (around 10^{-11}), showing excellent agreement between the ANN and BVP5C solutions. Whereas the number of epochs varies from 151 to 219 with β_1 , but the learning rate parameter μ stays constant at 10^{-10} . The very small gradients and low training times indicate a stable and efficient ANN convergence.

Fig. 2 presents a detailed comparison between the numerical BVP5C solution and the ANN-based solution for the velocity profile under varying velocity slip parameter β_1 . It has been revealed that by increasing the β_1 weakens the interaction between the fluid and the wall, reducing the momentum transferred from the boundary into the flow. As slip is enhanced, the near wall layers receive less shear, leading to a tremendous drop in fluid transport, overall velocity profile declines, and the momentum boundary layer becomes thinner. The first plot demonstrates that ANN predictions closely align with the BVP5C results for all β_1 values, pointing excellent

Table 1
Quantitative performance metrics for ANN solution.

β_1	Train Perf	Val Perf	Test Perf	RMSE	MAE	R2	Epochs	Mu	Gradient	TrainTime (s)
0.1	1.322e-11	1.272e-11	1.256e-11	3.612e-06	3.112e-06	1.0000	219	1.000e-10	9.781e-08	2.78
0.2	2.741e-11	2.611e-11	2.924e-11	5.243e-06	4.669e-06	1.0000	151	1.000e-10	9.957e-08	0.66
0.3	2.014e-11	2.156e-11	2.349e-11	4.567e-06	4.068e-06	1.0000	179	1.000e-10	9.950e-08	0.69

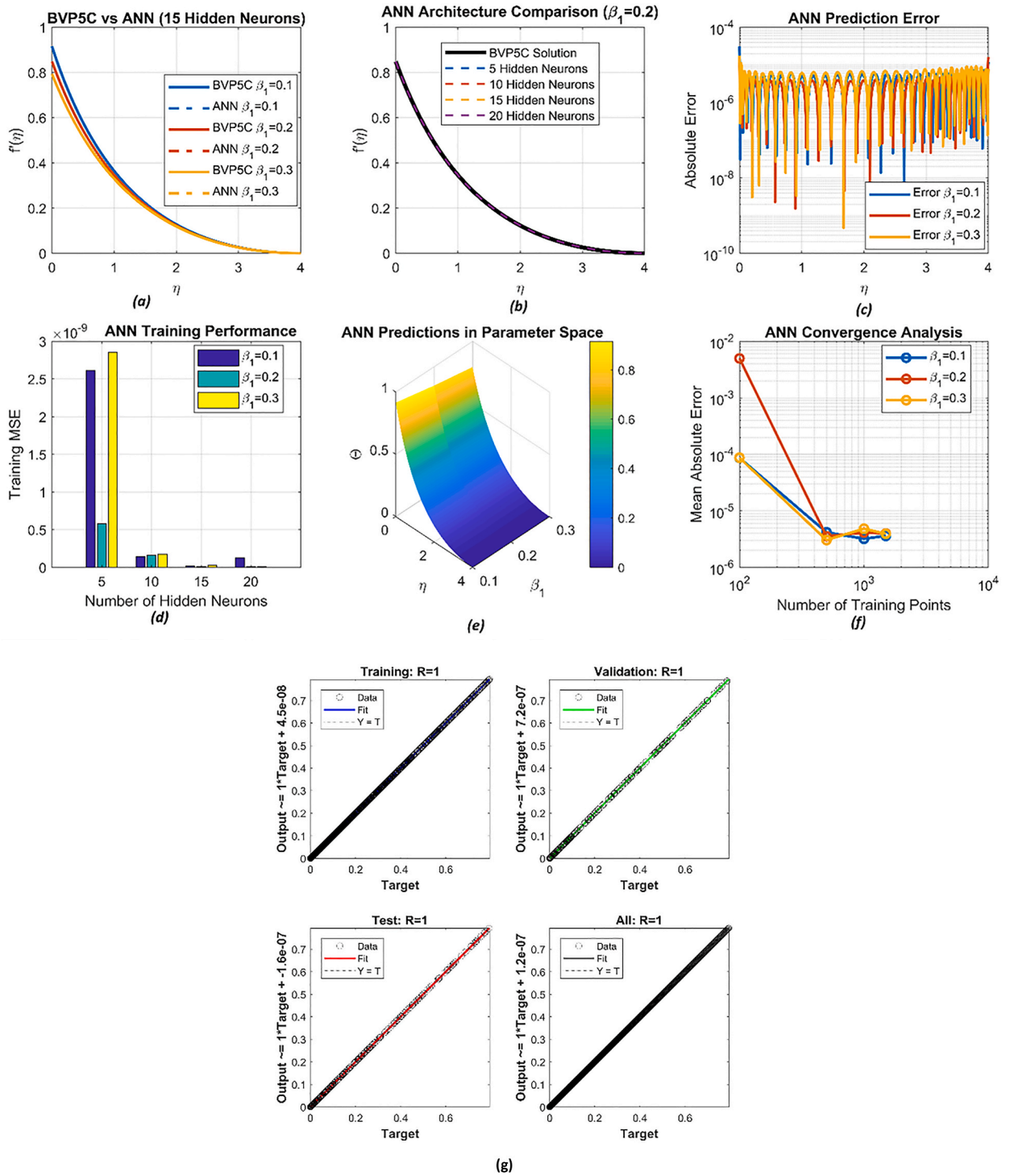


Fig. 2. Show analysis of BVP5C solution with ANN for velocity profile by varying velocity slip.

agreement. The architecture comparison as depicted in Fig. 2(b) show by increasing hidden neurons accuracy is improves substantially, with 15–20 neurons giving nearly identical results to BVP5C. Fig. 2(c) expresses the prediction error plot, exhibiting very low absolute errors (10^{-6} to 10^{-9}) for ANN scheme, exhibiting high precision. The training performance bar chart exhibited by Fig. 2(d) highlights that larger networks achieve smaller MSE, confirming better learning during training. Fig. 2(d) showing 3D parameter surface plot, illustrating the effect of β_1 versus η on the velocity field, displaying the ANN's ability to generalize in unseen regions. The

convergence analysis in Fig. 2(f) shows that errors are diminished as training points increase, stabilizing at near-optimal accuracy. The regression plots shown in Fig. 2(g) depict almost perfect agreement between ANN predictions and target values, with correlation coefficient $R = 1$ for training, validation, testing, and overall data. All data points are along the 45° line (Output = Target), revealing negligible prediction error. It is concluded that ANN model has excellent accuracy, reliability, and generalization capability for the given problem. The plots of Fig. 2 confirm that the ANN framework reliably approximates the BVP5C velocity solution while offering strong stability, low error, and excellent generalization.

Table No 2 shows all the key error metrics outcomes between the BVP5C and ANN solutions for the temperature profile $\theta(\eta)$ for varying thermal slip parameter, showing that the ANN achieves extremely low training, validation, and testing errors on the order of 10^{-10} , indicating excellent accuracy. It has been observed that as the values of β_2 is enhanced, the ANN requires fewer epochs and less training time, suggesting that the temperature profile becomes smoother and easier for the network to learn. The consistently small values of the gradient and fixed μ value confirms stable convergence and the attainment of an optimal solution.

An in-depth analysis for BVP5C solution with ANN for temperature profile is carried out and respective outcomes are presented in Fig. No. 3 for varying the thermal slip parameter. Fig. 3(a) shows that ANN accurately reproduces the BVP5C temperature curves for all β_2 values, confirming strong agreement. It has been observed that as β_2 increases, the temperature profile decreases due to reduced heat transfer from the wall, and both BVP5C and ANN capture this behavior consistently. The architecture comparison plot in Fig. 3(b) establishes that networks with 15 – 20 hidden neurons give the nearest match to the numerical solution. Fig. 3(c) shows very low absolute errors around ($10^{-6} - 10^{-9}$), indicating high prediction accuracy across the domain. Training performance bars depicted by Fig. 3(d) highlight that networks with more neurons achieve lower MSE, showing improved learning. The surface space plot in Fig. 3(e) illustrates how temperature varies with β_2 , confirming that ANN generalizes well beyond training points. The convergence plot in Fig. 3(f) shows rapid reduction in error as the number of training points increases, validating the stability and reliability of the ANN model for thermal-slip effects.

A comparison in table No 3 is presented between the BVP5C and ANN solutions for the temperature profile $\Phi(\eta)$ for varying β_3 values, ANN solution achieves extremely low training, validation, and testing errors on the order of $10^{-10} - 10^{-11}$, indicating excellent accuracy, the ANN requires fewer epochs and less training time for $\beta_3 = 0.3$, suggesting that the concentration profile becomes smoother and easier for the network to learn. The consistently small values of the gradient and fixed μ value confirms stable convergence and the attainment of an optimal solution. The regression plots shown in Fig. 3(g) depiction of perfect agreement between ANN predictions and target values, with correlation coefficient $R = 1$ for training, validation, testing, and overall data. This is the evidence ANN model has excellent accuracy, reliability, and generalization capability for the given problem.

Fig. 4 presented a detailed comparison between the BVP5C numerical solution and the ANN-based prediction for the concentration profile under different concentration slip values β_3 . Fig. 4(a) shows an absolute unanimity of both methodologies for varying concentration slip values which is depiction high accuracy. Fig. 4(b) evaluates solutions of ANN architectures, where it has been observed that increasing hidden neurons improves agreement with BVP5C. Very small absolute errors are observed in Fig. 4(c), typically below 10^{-4} , indicating strong reliability of the ANN model. Training performance in 4(d) highlights that higher slip values lead to larger training MSE, but still within adequate limits. The surface plot in 4(e) shows how concentration varies with β_3 , representing reduced concentration at higher slip. Fig. 4(f) show convergence analysis confirming that increasing training points significantly lowers the ANN error, ensuring model stability and robustness. Fig. 4(g) show ANN Performance metrics for different β_3 values it is reported that the MSE and RMSE values are very small ($\approx 10^{-6} - 10^{-3}$), indicating minimal prediction error of the ANN model. The coefficient of determination $R^2 = 1$ for all cases validates flawless fitness and excellent predictive accuracy. Training time is low ($\approx 0.24 - 0.30$ s), demonstrating that the model is computationally efficient and stable.

3.2. Temperature distribution

The contour plots in Fig. 5(a–f) show how temperature varies in the boundary layer under the influence of thermal slip β_2 , radiation parameter N_r , and Prandtl number Pr . It has been observed that with increasing β_2 the surface conduction is reduced, which weakens heat transfer and slightly lowers the fluid temperature near the wall. As the radiation parameter N_r increases in Fig. 5(c–d), additional radiative heat enhances energy transport, causing the temperature field to become warmer and more widely distributed. Higher value of N_r thus thickens the thermal boundary layer. In contrast, increasing the Prandtl number Pr as exhibited by Fig. 5(e–f), significantly reduces temperature because high Pr fluids (like oils) render stronger momentum diffusivity relative to thermal diffusivity. Due to this phenomenon the heat is more slowly diffused, and the thermal boundary layer is hampered.

Table 2
Data for comparison of BVP5C solution with ANN for $\theta(\eta)$

β_2	Train Performance	Validation Performance	Test Performance	RMSE	MAE	R2	Epochs	Mu	Gradient	Train Time (s)
0.1	3.058e-10	3.380e-10	3.724e-10	1.68808E-05	1.32964E-05	0.9999999	586	1.000e-08	9.970e-08	5.23
0.2	2.583e-10	2.635e-10	2.985e-10	1.75598E-05	1.37349E-05	0.9999999	519	1.000e-08	9.997e-08	1.75
0.3	2.833e-10	2.614e-10	3.381e-10	2.02393E-05	1.55159E-05	0.9999999	476	1.000e-08	9.920e-08	1.43

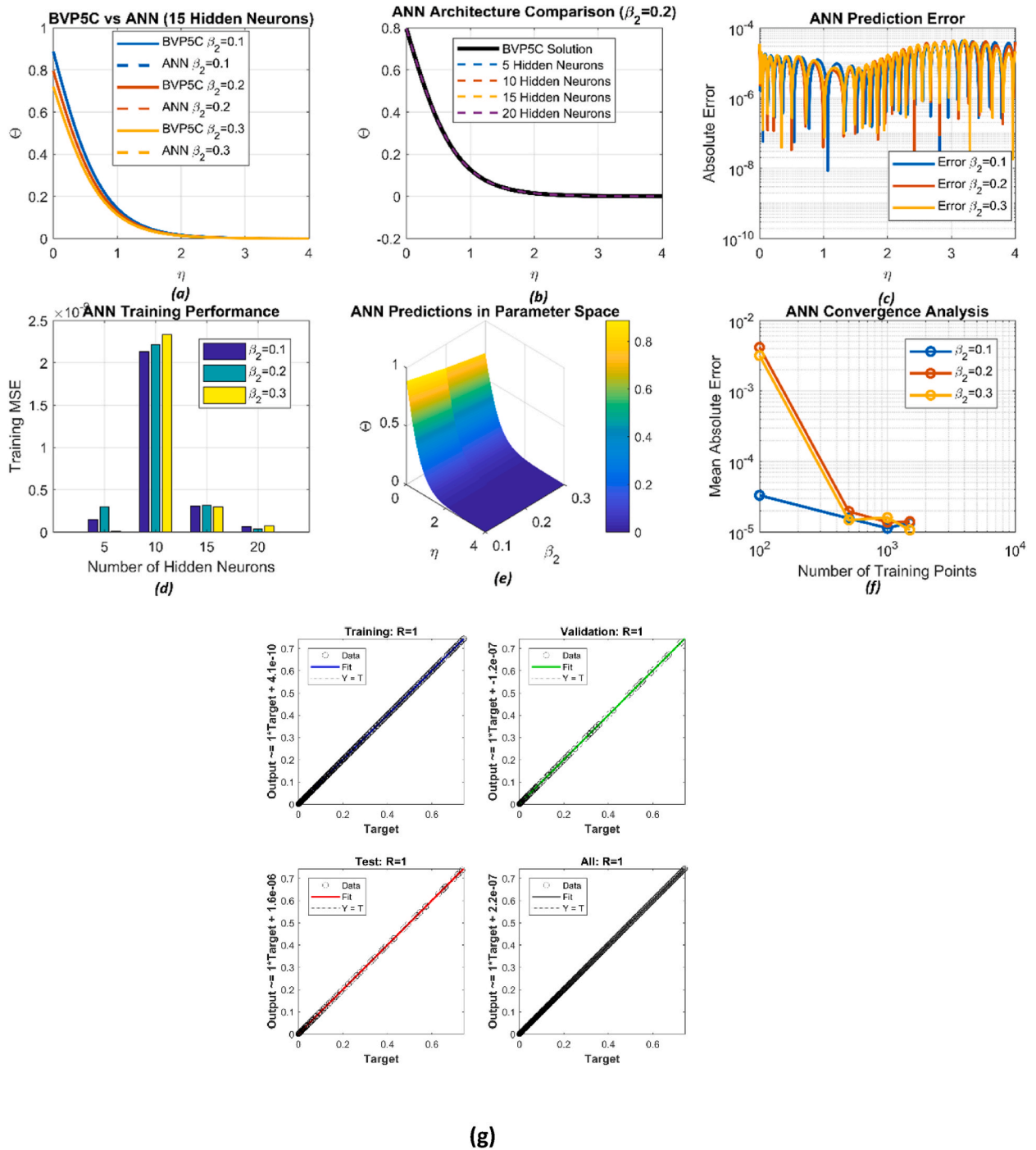


Fig. 3. Show analysis of BVP5C solution with ANN for temperature profile by varying thermal slip.

An increase in the Williamson parameter as shown in Fig. 6(a), reduces the effective viscosity of the non-Newtonian fluid, allowing stronger stretching and internal heat is produced, which ultimately raises the temperature profile.

Reductio in the resistance to deformation enhances thermal diffuse ion and thickens the thermal boundary layer.

Fig. 6(b) suggest that the curvature parameter slightly intensifies the temperature because curvature alters the flow dynamics, increasing local heat accumulation. It may be summed up that curvature effects produce a pronounced rise in temperature, while Williamson effects contribute a smaller but noticeable thermal enhancement. Fig. 6(c) shows decrease in $\Theta(\eta)$ along the boundary layer thickness, indicating heat diffusion from the surface to the fluid.

Table 3Data for comparison of BVP5C solution with ANN for $\Phi(\eta)$

β_3	Train_Perf	Val_Perf	Test_Perf	RMSE	MAE	R2	Epochs	Mu	Gradient	TrainTime_s
0.1	9.76922E-07	1.37451E-06	1.39202E-06	0.001048248	0.000890207	0.999977537	27	0.000001	3.05821E-05	0.1658729
0.2	7.38246E-07	7.40621E-07	9.47515E-07	0.000877492	0.000715986	0.999980179	27	0.0000001	6.643E-05	0.139486
0.3	8.71792E-07	1.17917E-06	1.44434E-06	0.001001888	0.000678609	0.999967572	21	0.0000001	0.000136635	0.1130615

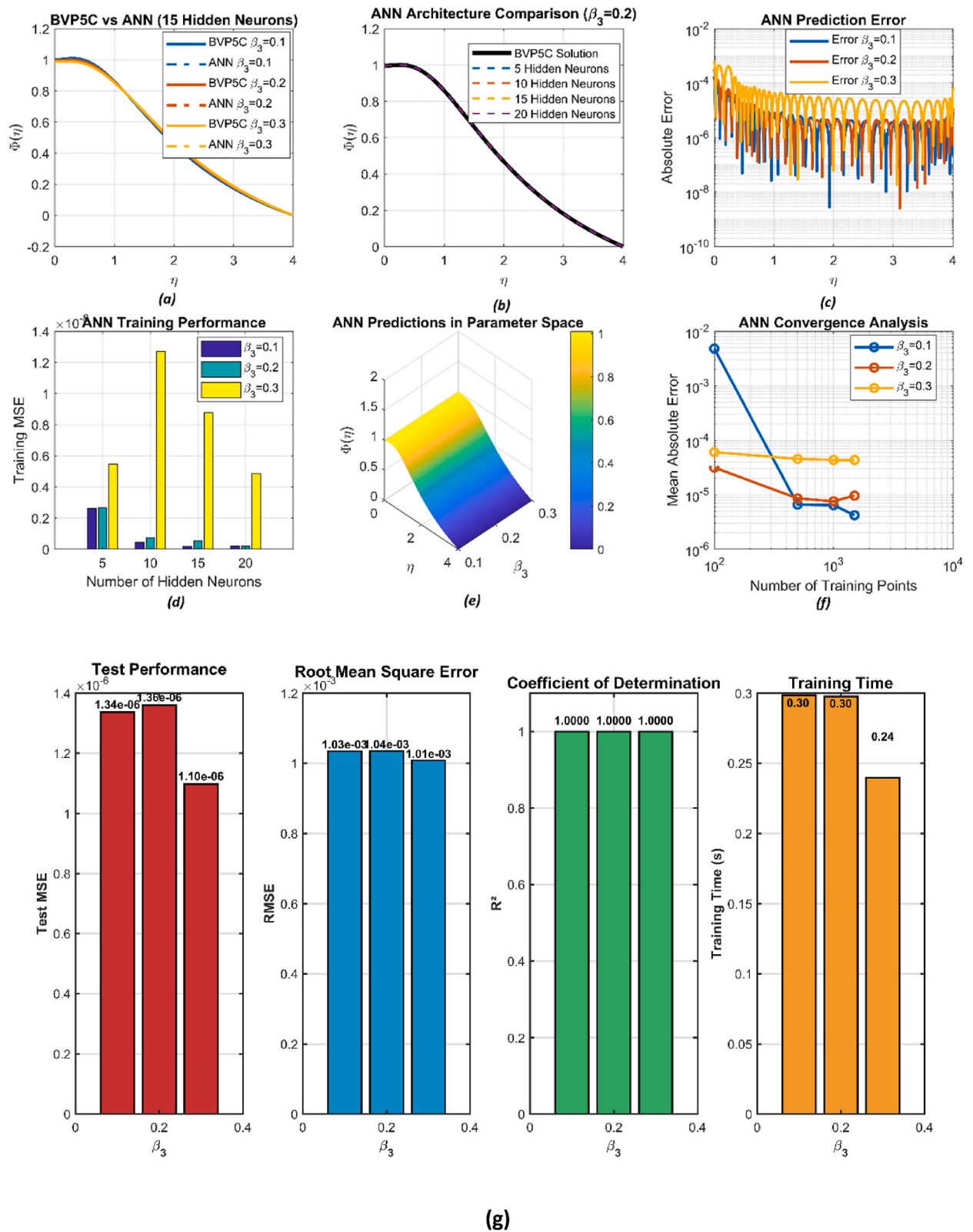


Fig. 4. Show analysis of BVP5C solution with ANN for concentration profile by varying concentration slip.

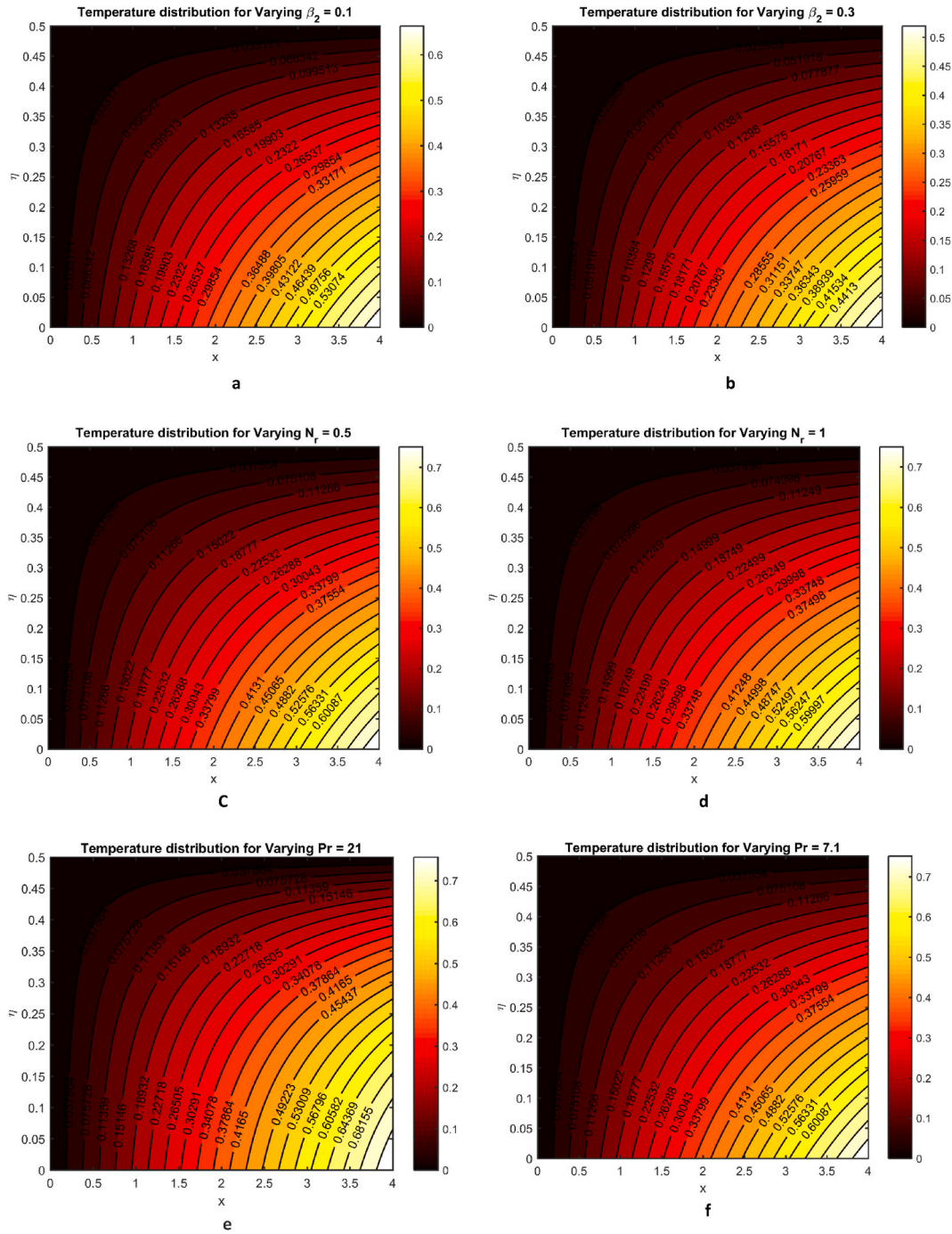


Fig. 5. Temperature distribution for thermal slip, radiation parameter, and Prandtl number.

An increase in the Prandtl number (Pr) corresponds to low thermal diffusivity, causing a thinner thermal boundary layer and a sharper drop in temperature. Lower Pr values permits heat to penetrate deeper into the fluid, therefore a slower decay. The parameter H_r slightly modifies the temperature distribution as observed in Fig. 6(d), where higher values improve heat transfer and slightly increase the temperature profile. In particular, both parameters control thermal transport, with Pr having a dominant effect and H_r producing comparatively weaker variations.

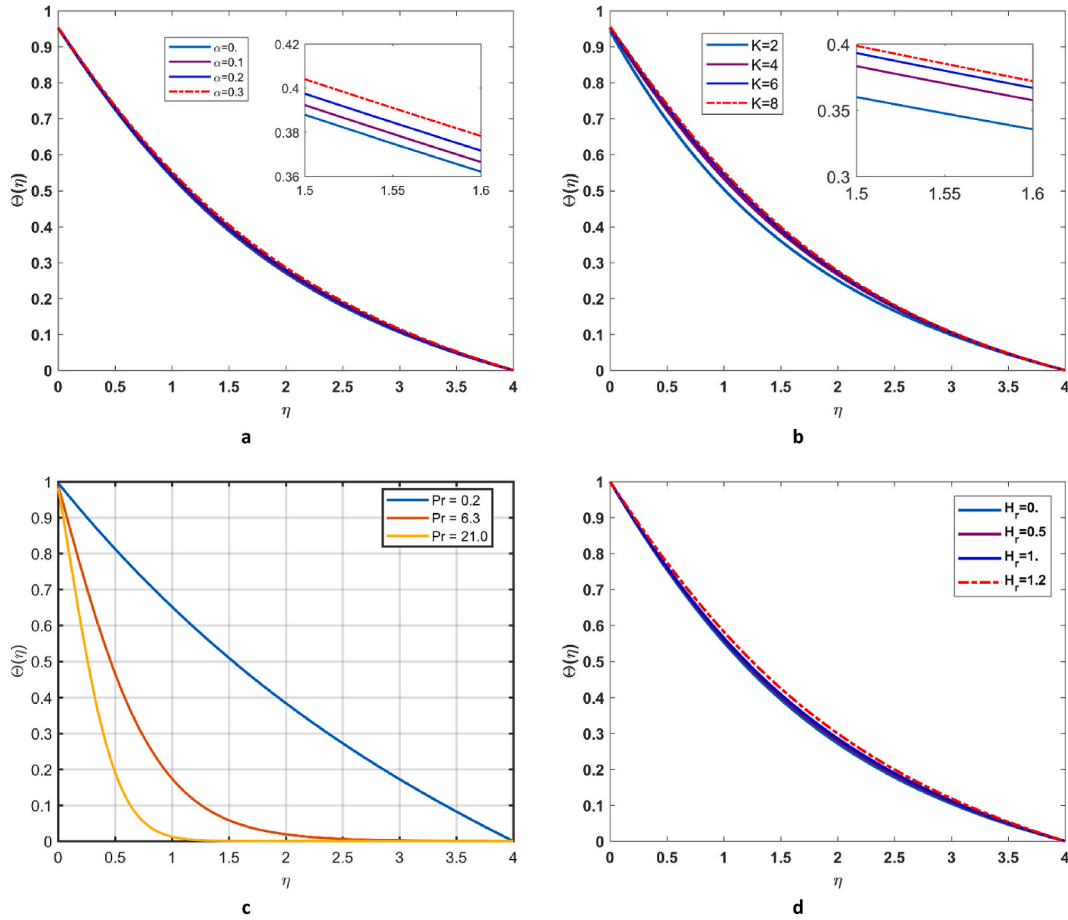


Fig. 6. Temperature profile for varying Williamson parameter and curvature parameter.

3.3. Fluid transport profile

The velocity contours for Williamson nanofluid using bioconvection for the swimming motion gyrotactic microorganisms in a stretched curved surface by varying velocity slip, the Weissenberg number, and the curvature parameter are shown in Fig. 7. It has been observed in Fig. 7(a–b) that by increasing the velocity slip parameter β_1 , reduction in wall shear resistance takes place, allowing the fluid to move more freely along the surface and increasing the overall velocity distribution. Fig. 7(c–d) show that higher W_e strengthens elastic effects in the Williamson nanofluid, which stretches the fluid molecules and enhances the flow rate, leading to a thicker velocity boundary layer. Further elastic deformation reduces viscous drag, causing higher velocity values throughout the domain. Fig. 7(e–f) show that by increasing bending or curvature of the stretched channel of reducing the constriction effect and allowing more fluid acceleration. This geometric influence produces a slight but noticeable rise in velocity contours.

3.4. Concentration profile

The concentration curves in Fig. 8 show how curvature K , magnetic field H_r , Brownian motion N_b , and Schmidt number S_c influence mass diffusion in the boundary layer. Increasing parameter K slightly enhances the stretching of the flow surface, allowing solute particles in Fig. 8(a) to diffuse more efficiently and producing a mild rise in concentration. Surge in the curvature parameter K enhances surface stretching, which boosts greater mass transport and accelerates the diffusion of solute particles into the fluid. Minor increase in concentration distribution due to improved mixing and reduced resistance to mass transfer near the surface. Fig. 8(b) showing that with increasing N_t decline in the concentration profile observed. An increase in the thermophoresis parameter drives nanoparticles away from the heated surface toward cooler regions, slowing down their accumulation near the boundary. Physically, this outward migration drops the concentration profile within the boundary layer, leading to a decline in overall concentration. Fig. 8 (c) suggests that the Brownian motion parameter N_b enhances random particle movement, which promotes molecular diffusion and therefore raises the concentration of nanofluid particles. On other hand Fig. 8(d) exhibits a higher Schmidt number S_c corresponds to fluids with weaker mass diffusivity, causing the solute to diffuse more slowly. Thereby concentration decreases sharply and the concentration boundary layer becomes thinner for large S_c .

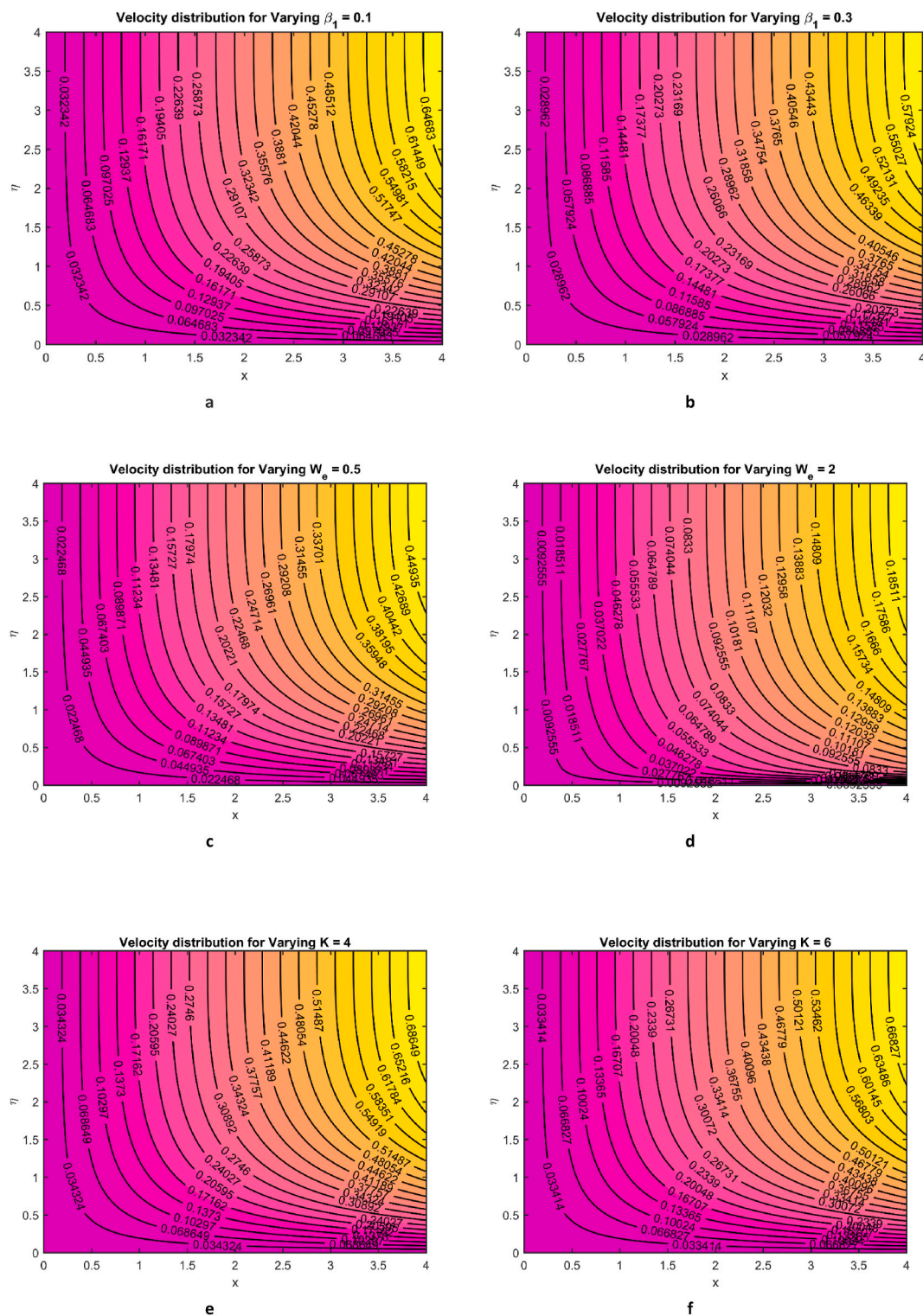


Fig. 7. Velocity distribution for velocity slip, Weissenberg number, and curvature parameter.

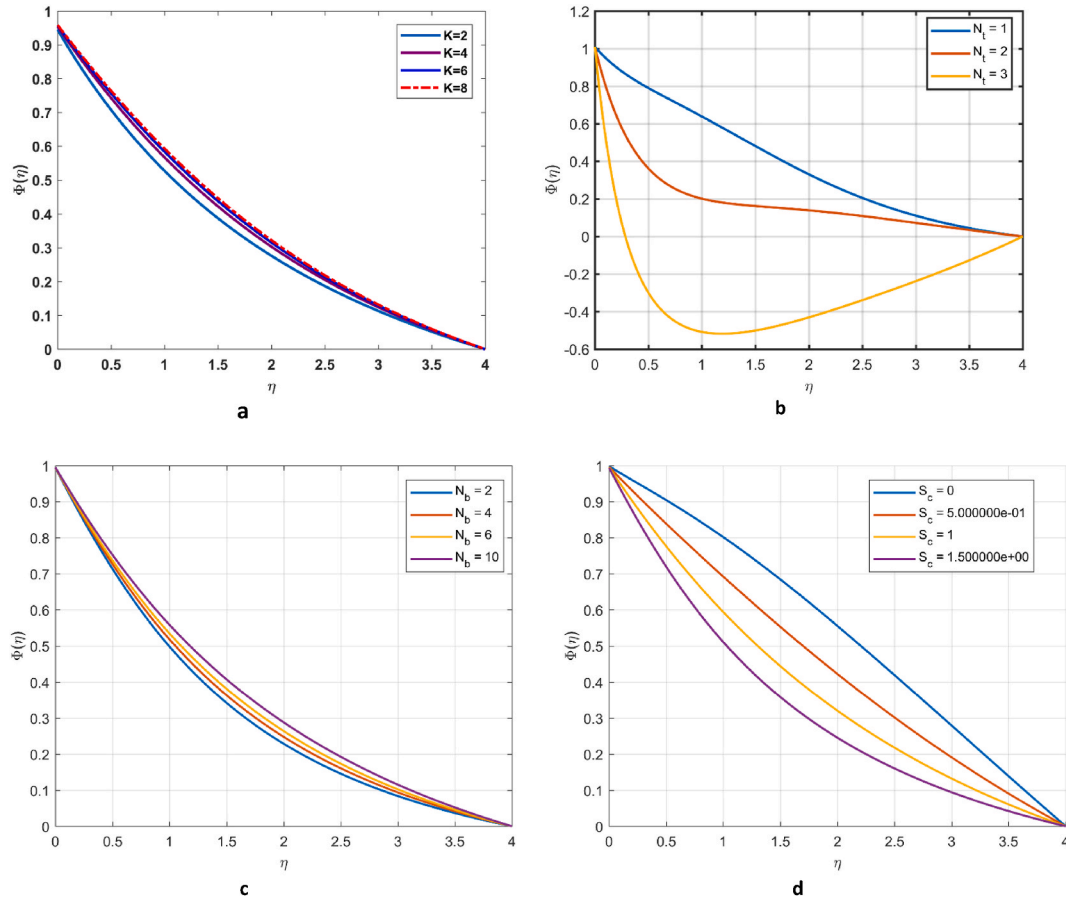


Fig. 8. Concentration profile by varying curvature parameter, Hartmann number, Brownian motion parameter and Schmidt number.

3.5. Gyrotactic microorganism profile

Gyrotaxis is defined to be microorganism's upward swimming against gravity due to density mismatch, developing bioconvection patterns in nanofluids. In the context of a curved stretching sheet, centrifugal forces from curvature modify their orientation and distribution within the boundary layer. Combined with nanoparticle transport, this plays significant role in enhancing mass/heat transfer in bio-microsystems while maintaining suspension stability through microorganism stirring motion. From Fig. 9(a) it is reported that higher curvature enhances fluid mixing, increasing microorganism concentration near the surface due to stronger vortices. Higher curvature strengthens shear and induces stronger rotational motion in the fluid, promoting bioconvective mixing. Resulting motile microorganisms to be more effectively transported and accumulate near the surface, increasing their local density and enhancing bioconvection effects. Magnetic field hinders flow motion as observed in Fig. 9(b), reducing microorganism transport through Lorentz force damping. The applied magnetic field generates a Lorentz force that opposes the fluid motion, increasing resistance and reducing velocity within the boundary layer. Weakens the convective transport, thereby limiting the movement and dispersion of microorganisms in the fluid. There has been substantial decline in microorganism concentration observed in Fig. 9(c) with bio-convection parameter. Surge in the bioconvection parameter builds up the collective motion of microorganisms, causing them to redistribute more rapidly away from the local region. Limits the accumulation near the surface, leading to a noticeable decline in microorganism concentration. An increase in the Péclet number indicates stronger convection in Fig. 9(d) relative to diffusion, causing microorganisms to be swept away more rapidly. This curbs their ability to disperse by molecular diffusion, which may lower microorganism concentration in the boundary layer. Thus, higher Péclet numbers results a sharper decline in the microorganism profile.

3.6. Skin friction, and other key parameters

Fig. 10 (a) demonstrates that as the Williamson fluid parameter α increases, the effective non-Newtonian resistance rises, which slows down fluid transport near-wall and thus slightly reduces skin friction. On other by increasing the Weissenberg number W_e as shown in Fig. 10(b) strengthens the elastic effects of the fluid, allowing it to stretch and resist deformation more strongly, leading to a

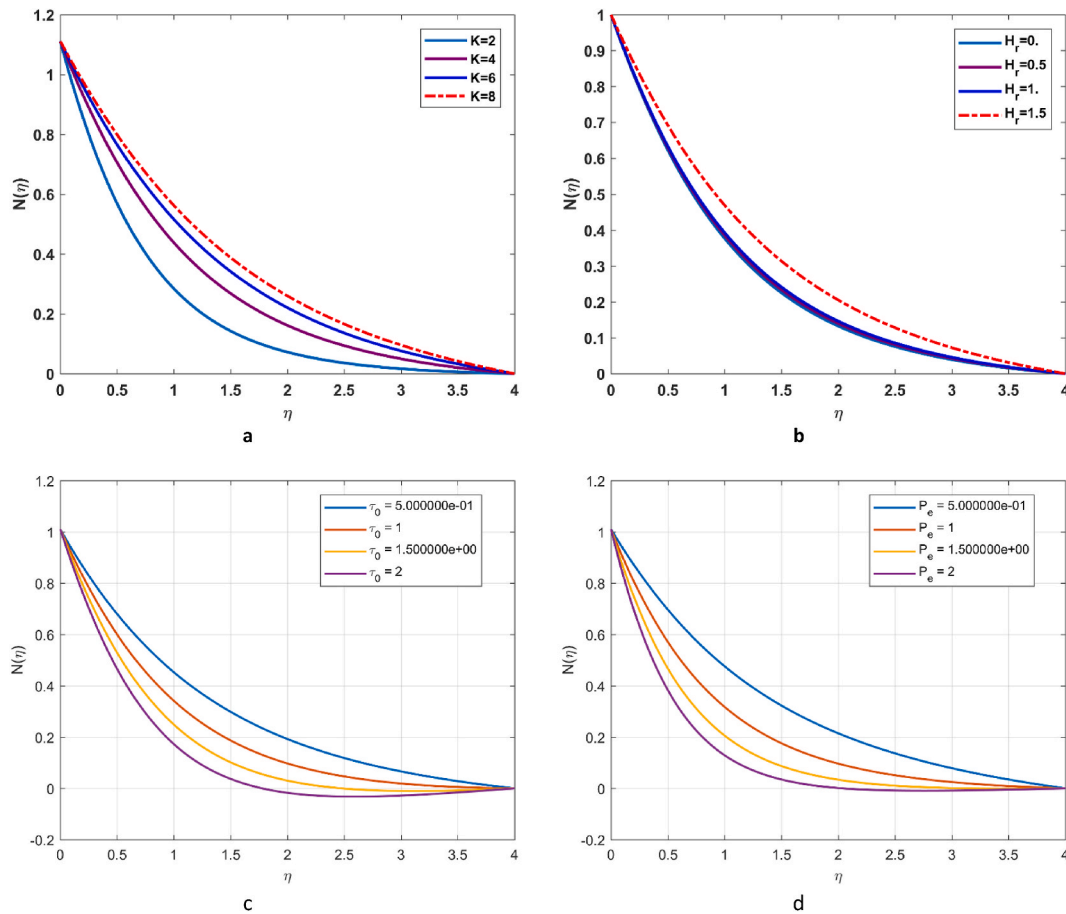


Fig. 9. Motile microorganism profile by varying curvature parameter, Hartmann number, bioconvection parameter and Pecklet number.

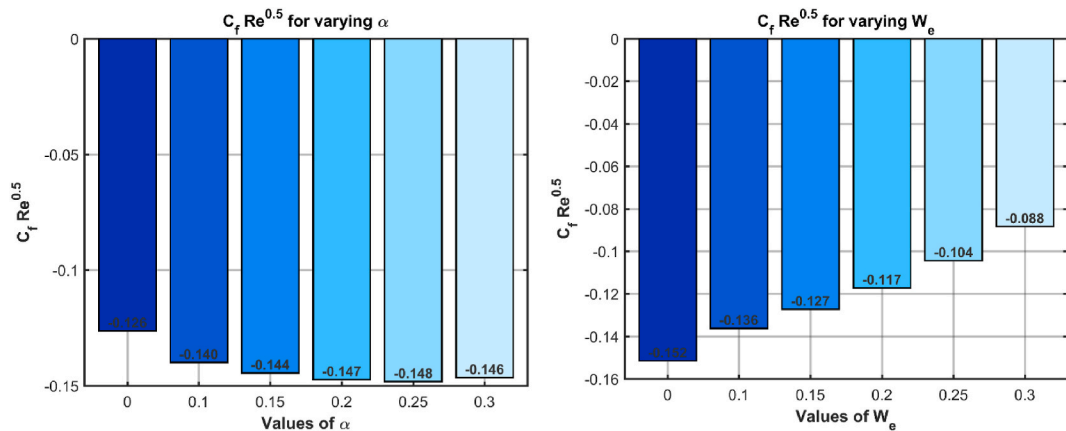


Fig. 10. Showing skin friction plots for varying Williamson fluid parameter, and Weissenberg number.

marked drop in wall shear stress. Lower $C_f Re^{0.5}$ values indicate weaker momentum transfer from fluid to the wall as both α and W_e grow. Stronger non-Newtonian and elastic behavior make the flow more resistant to shearing, hence decreasing skin friction.

It is quite evident from Table No 4 that as α enhanced, the thermal slip at the wall grows, weakening the temperature gradient at the surface and slightly reducing the heat-transfer rate (Nu). Surge in the thermal radiation, boosts energy transport away from the wall, causing a rise in Nu . A higher curvature parameter K strengthens the geometric stretching of the flow surface, improve fluid transport and result rise in heat-transfer rates. Thermal slip reduces, while radiation and curvature enhance heat transfer, as depicted in the

Table 4
Nusselt number for varying α , N_r and K .

α	N_r	K	$\frac{Nu}{\sqrt{Re}}$
0.1			0.33632557
0.2			0.334371363
0.3			0.335369871
	0.1		0.33632557
	0.2		0.36596359
	0.3		0.395577608
		0.4	0.33632557
		0.5	0.34683522
		0.6	0.355054291
		0.7	0.361678396

Table 5
Sherwood number for varying P_e , S_c and S_b .

P_e	S_c	S_b	$\frac{S_h}{\sqrt{Re}}$
1			0.331210319
1.4			0.331210296
1.8			0.331210295
	1		0.331210323
	1.4		0.355298664
	1.8		0.368674878
	1.3		0.350323022
	1.4		0.355298664
	1.5		0.359600102
	1.6		0.363247453
	1.7		0.366263868
	1.8		0.368674878
	1.9		0.370507783
	2		0.371791132
		1	0.331210323
		1.4	0.331210323
		1.8	0.331210323

Table 6
density of the motile organism for varying P_e , S_c and S_b .

P_e	S_c	S_b	$\frac{M_n}{\sqrt{Re}}$
1			0.382464556
1.4			0.368750306
1.8			0.350016084
	1		0.389785543
	1.4		0.381837755
	1.8		0.371192141
		1	0.371187977
		1.4	0.380543433
		1.8	0.387330423

increasing values of $\frac{Nu}{\sqrt{Re}}$

Table No 5 shows that when the value of Péclet number, strength in convection relative to diffusion which enhances mass transport toward the wall and therefore slightly raises the Sherwood number. The S_c represents the ratio of momentum to mass diffusivity, higher S_c means weaker species diffusion, leading to sharper concentration gradients and a significant rise in S_h , as seen in the table. The bioconvective Schmidt number (S_b) characterizes microorganism diffusivity, increasing S_b reduces microbial diffusion and steepens the concentration boundary layer, thus increasing the S_h . Together, P_e , S_c , and S_b collectively intensify concentration gradients at the wall, which directly enhances mass transfer rates.

The motile microorganism density decreases with increasing P_e as observed in Table No 6, because stronger advective transport sweeps microorganisms away from the boundary, reducing their accumulation. Surge in the S_c enhances the resistance to mass diffusion, which causes microorganisms to cluster more strongly near the wall, thereby increasing their density gradient. Higher S_b suppresses microorganism diffusion and strengthens bioconvective trapping, leading to an overall rise in the motile microorganism density near the surface.

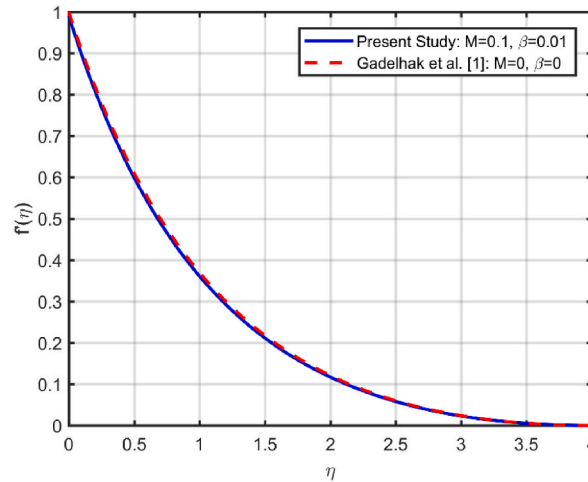


Fig. 11. Comparison with previously published results.

Validation with previous result.

Fig. 11 compares the present study ($M = 0.5, \beta = 0.01$) with Gadelhak et al. [1]'s work ($M = 0, \beta = 0$). An excellent agreement of the present study with Gadelhak et al. [1] is observed confirming the authenticity and validity of reported investigation. The close overlap between the present solution and the published results demonstrates excellent agreement, confirming the accuracy and reliability of the numerical formulation. Showing that the validation is not limited solely to ANN predictions but is also benchmarked against established literature. Hence, the model is verified both through ANN approximation and consistency with existing published results, strengthening the credibility of the study.

4. Conclusion

A mathematical model with comprehensive key physical deliberation is presented. A steady incompressible transport of MHD Williamson nanofluid with multiple slip effects, heat, mass transfer and with gyrotactic microorganisms is focused in a linear curved stretching sheet. The analyze the flow scheme curvilinear coordinates system is Velocity, thermal, concentration, and microorganism slips are applied with thermophoresis, Brownian motion, gyrotactic microorganisms, and thermal radiation effects. The formulation composed of PDES, which are subjected to similarity transformations for simplification into ODEs. The solution for the radiative induced nanofluid is explored with BVP5C solution which is validated using ANN based Levenberg Marquardt backpropagation. The findings of the study may be summarized as:

- The ANN model showed outstanding accuracy and reliability, as evidenced by extremely small training, validation, and testing errors ($\approx 10^{-11}$), demonstrating excellent agreement with the BVP5C numerical solution.
- The ANN training process is stable and efficient, indicated by a consistently small learning rate ($\mu = 10^{-10}$), very low gradients, and a narrow range of required epochs, confirming smooth convergence and robustness of the neural network approach.
- It has been observed that the thermal radiation plays a dominant role in heating the fluid for the curved stretched surface, it significantly gives impetus to the temperature and enlarges the thermal boundary layer, indicating enhanced energy transport through radiative effects.
- A noticeable reduction in temperature and a thinner thermal boundary layer is observed with higher of Prandtl number values, confirming that thermal diffusivity decreases with Higher Prandtl number.
- It has been revealed that velocity slip and Weissenberg number significantly enhance fluid motion, as reduced shear resistance and stronger elastic effects allow the Williamson nanofluid to accelerate more freely, resulting in higher velocity and a thicker momentum boundary layer.
- Rising values of curvature parameter accelerates the fluid transport, revealing that geometric effects play a supportive role by reducing flow constriction and slightly elevating the overall velocity distribution in the bioconvective nanofluid system.
- The diffusion phenomenon is enhanced with curvature and Brownian motion N_b and is lowered with magnetic field and high Schmidt number, the concentration boundary layer becomes thicker when diffusion dominates and significantly thinner when mass diffusivity weakens.
- The curvature, magnetic field, bioconvection strength, and Péclet number have significantly influenced microorganism transport: curvature enhances fluid mixing and promotes distribution, whereas magnetic and bioconvective effects tend to suppress dispersion and limit microorganism mobility.
- Higher Péclet numbers indicate stronger convective transport, which rapidly carries microorganisms away from the surface, resulting in a pronounced reduction in their concentration within the boundary layer.

- > Increasing the Williamson fluid parameter and the Weissenberg number strengthens non-Newtonian and elastic effects, which significantly weaken wall shear stress and reduce skin friction.
- > Decline in the in skin friction profile confirms that stronger rheological resistance limits momentum transfer to the wall, resulting in lower shear and more resistant fluid motion.

Reported findings are highly useful in diverse engineering applications such as polymer processing, coating flows, and lubrication systems where lower shear at the wall improves efficiency. However, non-Newtonian and elastic effects may also introduce flow instability or reduced heat transfer, limiting their applicability in systems having thermal control. The model assumes idealized boundary conditions and a simplified rheological structure, which may not fully represent real industrial fluids. Future investigations may incorporate temperature-dependent viscosity, variable material properties, and experimental validation to enhance practical relevance. Additionally, extending the analysis to 3D, transient, or turbulent regimes would provide deeper insight into real-world fluid behavior.

CRediT authorship contribution statement

Ali Imran: Writing – original draft, Software, Investigation, Data curation, Conceptualization. **Hafiza Elbadie:** Writing – review & editing, Validation, Funding acquisition, Data curation. **Jongsuk Ro:** Writing – review & editing, Visualization, Resources, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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