

SOME REMARKS ON STABLE MINIMAL SURFACES IN THE CRITICAL POINT OF THE TOTAL SCALAR CURVATURE

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ABSTRACT. It is well known that critical points of the total scalar curvature functional $\mathcal{S}$ on the space of all smooth Riemannian structures of volume 1 on a compact manifold M are exactly the Einstein metrics. When the domain of $\mathcal{S}$ is restricted to the space of constant scalar curvature metrics, there has been a conjecture that a critical point is isometric to a standard sphere. In this paper we investigate the relationship between the first Betti number and stable minimal surfaces, and study the analytic properties of stable minimal surfaces in M for $n = 3$.

1. Introduction

Let M^n be a n -dimensional compact manifold and $\mathcal{M}_1$ be the set of smooth Riemannian metrics on M^n of volume 1. Given a metric $g \in \mathcal{M}_1$, the total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \rightarrow \mathbb{R}$ is defined by

$$(1) \quad \mathcal{S}(g) = \int_{M^n} s_g dv_g,$$

where s_g is the scalar curvature of g and dv_g the volume form determined by the metric and orientation.

The importance of (1) lies in the fact that critical points of $\mathcal{S}|_{\mathcal{M}_1}$ are Einstein. In order to produce Einstein metrics by variational method, we may adopt a min-max scheme whereby one first minimizes $\mathcal{S}$ in each conformal class, then maximizes over all conformal classes. The constrained problem obtained by restricting $\mathcal{S}$ to a conformal class is known as the Yamabe problem and is always solvable. In 1987, Besse proposed a conjecture in the book [1] that the “max” part is an Einstein metric. In fact, the conjecture in [1] states that critical points of the total scalar curvature on the set of constant scalar curvature are Einstein.

Received May 23, 2008.

2000 *Mathematics Subject Classification.* 58E11, 53C25.

Key words and phrases. total scalar curvature, critical points, stable minimal surfaces.

This Research was supported by the Chung-Ang University Research Grants in 2006.

A critical point g of this restricted variational problem satisfies the following equation:

$$(2) \quad z_g = s'_g(f),$$

where z_g is the traceless Ricci tensor, f is a function on M^n with vanishing mean value, and

$$s'_g(f) = D_g df - (\Delta_g f)g - f r_g,$$

where r_g is the Ricci curvature of g .

J. Lafontaine showed that if a solution metric g of (2) is conformally flat, such a metric is Einstein (see [9]). The author showed that if a solution function f of (2) has its minimum greater than or equal to -1 , g is Einstein (see [4]). Also it was shown in [8] that for $n = 3$, if $\ker s'_g \neq 0$ and $H_2(M^3, \mathbb{Z}) = 0$, then g is Einstein. The geometric structure of an Einstein solution is known to be simple due to M. Obata, who showed that such a solution is isometric to a standard n -sphere (see [11]).

Restricting our considerations to $n = 3$, the purpose of the present paper is to prove the following two results. First we have the following.

Theorem 1.1. *Let (g, f) be a non-trivial solution of (2) on a 3-dimensional compact oriented manifold M^3 . If $\ker s'_g \neq 0$, then the first Betti number vanishes if and only if there is no embedded compact oriented stable minimal surface in M^3 .*

Theorem 1.1 clarifies the relationship between the first Betti number and the stable minimal surface in dimension 3. Secondly, we have the following technical result concerning about the structure of stable minimal surfaces in M^3 .

Theorem 1.2. *Let Σ be a stable minimal surface in M^3 and $\rho = z(\nu, \nu)$ for a normal vector field ν on Σ . Then $\nu(\rho) = 0$ and $\nu\nu(\rho) > 0$ on Σ .*

Theorem 1.2 is one of keys to investigate the existence of stable minimal surfaces and will be used in our forthcoming papers. In Section 2, Theorem 1.1 will be proved. We shall study the analytic properties of stable minimal surfaces in M^3 and prove Theorem 1.2 in Section 3.

2. The first Betti number and stable minimal surfaces

In [5], we claimed that $H_2(M^3, \mathbb{Z}) = 0$ implies the non-existence of embedded compact oriented stable minimal surface Σ in M^3 . In fact, it turns out to be true. However, the proof of this claim was not clear. Here we present the proof of Theorem 1.1, which makes the relationship between the first Betti number and stable minimal surfaces in M^3 clear.

Proof. It is well known that each element in $H_{n-1}(M^n, \mathbb{Z})$ can be represented by sums of embedded compact oriented stable minimal hypersurfaces for $3 \leq$

$n \leq 7$ (see [10]). Therefore, it is clear that if there is no stable minimal surface in M^3 , then $H_2(M^3, \mathbb{Z}) = 0 = b_1(M^3)$.

On the other hand, first we shall show that a stable minimal surface Σ in M^3 would be homeomorphic to S^2 . Let φ be a function in $\ker s_g'^*$ and $\Gamma = \varphi^{-1}(0)$. Thus

$$(3) \quad s_g'^*(\varphi) = 0.$$

It is known that Γ has no critical point (see [2]). Therefore Γ is a union of surfaces. By (3), it is easy to check that Γ is totally geodesic. It was proved in [6] that every compact oriented stable minimal hypersurface in M^n should be contained in Γ . This statement is true for arbitrary dimension n . Therefore, Σ coincides with a connected component of Γ , and is totally geodesic. Then, by the Gauss-Codazzi equation, the intrinsic Gauss curvature of Σ is given by

$$(4) \quad K_\Sigma = \frac{s}{6} - z(\nu, \nu),$$

where ν is a normal vector field on Σ . Let $\rho = z(\nu, \nu)$ on Σ . It was proved in [8] that $(1+f)^3\rho$ is constant on each components of Γ . Let $(1+f)^3\rho = c$ for some constant c on Σ . From the following relation

$$(5) \quad (1+f)\rho = -\Delta'f - \frac{s}{3}f$$

on Σ and the fact in [4] that a stable minimal surface should be located in $M_0 = \{x \in M \mid f(x) < -1\}$, we have

$$c \int_\Sigma \frac{1}{(1+f)^2} = \int_\Sigma (1+f)\rho = -\frac{s}{3} \int_\Sigma f > 0.$$

Thus $c > 0$, implying that $\rho < 0$ on Σ . It follows that $K_\Sigma > 0$, implying that Σ is homeomorphic to S^2 .

It is well known (see, for example, [3]) that if $b_1(M^3) = 0$, then a surface homeomorphic to S^2 separates M^3 . Thus Σ separates M^3 into two pieces. However, by the equation (15) in [5], we get

$$0 = \int_\Sigma f |d\varphi|,$$

which is impossible since $f < -1$ on Σ . This contradiction leads to the proof of Theorem 1.1. $\square$

3. Structures of Σ and analytic properties of $z(\nu, \nu)$

We consider

$$\begin{aligned} \Sigma^A &= \{x \in \Sigma \mid df \text{ is parallel to } \nu \text{ with } df \neq 0 \text{ at } x\}, \\ \Sigma^B &= \{x \in \Sigma \mid df \text{ is not parallel to } \nu \text{ with } df \neq 0 \text{ at } x\}, \\ \Sigma^C &= \{x \in \Sigma \mid df(x) = 0\}. \end{aligned}$$

Note $\langle df, \nu \rangle$ is constant by Lemma 5 in [5]. Thus, from the fact that $\langle df, \nu \rangle \neq 0$ on Σ^A and $\langle df, \nu \rangle = 0$ on Σ^C , and Σ is connected, either $\Sigma = \Sigma^A \cup \Sigma^B$ or $\Sigma =$

$\Sigma^B \cup \Sigma^C$ holds. For the case when $\Sigma = \Sigma^A \cup \Sigma^B$, we shall prove in Lemma 3.1 that there exists a function $\tilde{f}$ such that $d\tilde{f} = 0$ on Σ^A and $d\tilde{f}$ is tangent to Σ^B . In other words, $\Sigma^B = \{x \in \Sigma \mid d\tilde{f} \text{ is tangent to } \Sigma \text{ with } d\tilde{f} \neq 0 \text{ at } x\}$. For the case when $\Sigma = \Sigma^B \cup \Sigma^C$, $d\tilde{f}$ is tangent to Σ^B since $e = \langle df, \nu \rangle$ is constant on Σ and $e = 0$ on Σ^C . In other words, $\Sigma^B = \{x \in \Sigma \mid d\tilde{f} \text{ is tangent to } \Sigma \text{ with } d\tilde{f} \neq 0 \text{ at } x\}$. In other words, both cases are essentially the same. Therefore we shall consider the first case only.

In order to prove Theorem 1.2, we need the following two lemmas.

Lemma 3.1. *There is no open set in Σ^A .*

Proof. Assume the contrary. Let Ω be an open set in Σ^A . Take a point p in Ω . In virtue of Lemma 5 in [5], stating that $\Phi = |d\varphi|^2$ and $\eta = \langle df, d\varphi \rangle$ are constant along Σ , we define $\tilde{f}$ by $\tilde{f} = f - \Phi_0^{-1}\eta_0\varphi$, where $\Phi_0 = \Phi|_\Sigma$ and $\eta_0 = \eta|_\Sigma$, and $\tilde{W}$ by $\tilde{W}^{\frac{1}{2}} = |d\tilde{f}| \geq 0$. Then $\nu = W^{-\frac{1}{2}}df = \Phi^{-\frac{1}{2}}d\varphi$ and $d\tilde{f} = df - \langle df, \nu \rangle\nu = 0$ on Σ^A . In particular, $\tilde{W}^{\frac{1}{2}}(p) = 0$, and $\nu(\tilde{W}^{\frac{1}{2}})(p) = 0$ if $\nu(\tilde{W}^{\frac{1}{2}})$ exists at p . We shall prove that

$$(6) \quad \nu(\tilde{W}^{\frac{1}{2}})(p) \neq 0,$$

which is a contradiction. This contradiction completes the proof of Lemma 3.1. Let U be a small neighborhood of p in M . Note that $\tilde{W} = W - 2a\eta + a^2\Phi$ with $a = \Phi_0^{-1}\eta_0$. On $U \setminus (\Sigma \cap U)$,

$$(7) \quad \nu(\tilde{W}^{\frac{1}{2}}) = \frac{\nu(\tilde{W})}{2\tilde{W}^{\frac{1}{2}}} = \frac{\nu(W) - 2a\nu(\eta) + a^2\nu(\Phi)}{2\sqrt{W - 2a\eta + a^2\Phi}}.$$

The numerator of (7) can be written as

$$\begin{aligned} & 2\langle D_\nu df, df \rangle - 2a(\langle D_\nu df, d\varphi \rangle + \langle df, D_\nu d\varphi \rangle) + 2a^2\langle D_\nu d\varphi, d\varphi \rangle \\ &= 2W^{\frac{1}{2}}(h\rho - \frac{s\tilde{f}}{6}) - 2a(\Phi^{\frac{1}{2}}(h\rho - \frac{s\tilde{f}}{6}) + W^{\frac{1}{2}}(\varphi\rho - \frac{s\varphi}{6})) + 2a^2\Phi^{\frac{1}{2}}(\varphi\rho - \frac{s\varphi}{6}) \\ &= 2(W^{\frac{1}{2}} - a\Phi^{\frac{1}{2}})(\rho\tilde{h} - \frac{s}{6}\tilde{f}), \end{aligned}$$

where $\tilde{h} = 1 + \tilde{f}$. Thus

$$\nu(\tilde{W}^{\frac{1}{2}}) = \frac{W^{\frac{1}{2}} - a\Phi^{\frac{1}{2}}}{\tilde{W}^{\frac{1}{2}}}(\rho\tilde{h} - \frac{s}{6}\tilde{f}).$$

Since df is parallel to ν at p , $\eta(x) = \langle df, d\varphi \rangle(x)$ goes to $|df|(p)|d\varphi|(p) = W^{\frac{1}{2}}(p)\Phi^{\frac{1}{2}}(p)$ as x goes to p . Therefore

$$\frac{(W^{\frac{1}{2}} - a\Phi^{\frac{1}{2}})^2}{\tilde{W}}(x) = \frac{W - 2aW^{\frac{1}{2}}\Phi^{\frac{1}{2}} + a^2\Phi}{W - 2a\eta + a^2\Phi}(x) \rightarrow 1$$

as x goes to p , which implies the existence of $\nu(\tilde{W}^{\frac{1}{2}})$. In particular, by (2),

$$(8) \quad \lim_{d(x,p) \rightarrow 0} \nu(\tilde{W}^{\frac{1}{2}})(x) = \lim_{d(x,p) \rightarrow 0} \langle D_\nu d\tilde{f}, \nu \rangle(x) = (\rho\tilde{h} - \frac{s}{6}\tilde{f})(p).$$

From the assumption of Ω in Σ^A , $\Delta' \tilde{f} = 0$ on Ω since $\tilde{f}$ is constant. By (5) we have

$$(9) \quad \tilde{h}\rho = -\frac{s}{3}\tilde{f}$$

on Ω . Substitution of (9) and the fact that $f < -1$ on Σ into (8) gives

$$\nu(\tilde{W}^{\frac{1}{2}})(p) = (\rho\tilde{h} - \frac{s}{6}\tilde{f})(p) = -\frac{s}{2}\tilde{f}(p) = -\frac{s}{2}f(p) > \frac{s}{2}.$$

□

The behavior of ρ on Σ^B is given as follows:

Lemma 3.2. *On Σ^B , we have*

$$(10) \quad \nu\nu(\rho) = \left(\frac{s}{2h} - \frac{6W}{h^2} \right) \rho.$$

In virtue of Lemma 3.1, we may conclude that the equation (10) holds for all of Σ . In the proof of Theorem 1.1, it was proved that $\rho < 0$ on Σ . Hence, by (10), $\nu\nu(\rho)$ should be positive on Σ , completing the proof of Theorem 1.2.

What remains is the proof of Lemma 3.2, which will be the main focus of the rest of this section. Let Σ_ϵ be $\Sigma_\epsilon = \varphi^{-1}(\epsilon) \cap U_\epsilon$ for a connected neighborhood U_ϵ of Σ . Let ν be the unit normal vector field on Σ_ϵ , and let $\Psi^{\frac{1}{2}} = |d\tilde{f}^\top|$, where

$$(11) \quad d\tilde{f}^\top = d\tilde{f} - \langle d\tilde{f}, \nu \rangle \nu.$$

Then $\Psi^{\frac{1}{2}}|_\Sigma = |d\tilde{f}| = \tilde{W}^{\frac{1}{2}}$. Also let $Y = \frac{d\tilde{f}^\top}{|d\tilde{f}^\top|}$ on U_ϵ , X be the unit tangent vector field to Σ which is orthogonal to Y . Then $\{X, Y, \nu\}$ is an orthonormal frame field on U_ϵ with $Y|_\Sigma = \frac{d\tilde{f}}{|d\tilde{f}|}$.

From now on, we will abuse notation and write f for $\tilde{f}$ and W for $\tilde{W}$. With these setting, we proved in [8] that $h^3\rho$ is constant on Σ as mentioned in Section 2. In particular, on Σ^B

$$(12) \quad X(\rho) = 0, \quad Y(\rho) = -\frac{3}{h}W^{\frac{1}{2}}\rho.$$

Note that, on Σ^B

$$(13) \quad D_X\nu = \Phi^{-\frac{1}{2}}Dd\varphi(X, \cdot) = 0,$$

$$(14) \quad D_Y\nu = \Phi^{-\frac{1}{2}}Dd\varphi(Y, \cdot) = 0,$$

$$(15) \quad D_\nu(df)^\top = 0$$

since $D_\nu(df)^\top = D_\nu df - \langle D_\nu df, \nu \rangle \nu$. Therefore

$$(16) \quad D_\nu X = \langle D_\nu X, Y \rangle Y = -\frac{1}{W^{\frac{1}{2}}} \langle X, D_\nu(df)^\top \rangle Y = 0,$$

$$(17) \quad D_\nu Y = \langle D_\nu Y, X \rangle X = W^{-\frac{1}{2}} \langle D_\nu(df)^\top, X \rangle = 0.$$

Since $D_X(df)^\top = D_X df$ and $z(X, X) = z(Y, Y) = -\frac{1}{2}\rho$,

$$\begin{aligned} D_X X &= \langle D_X X, Y \rangle Y + \langle D_X X, \nu \rangle \nu = -\langle X, D_X Y \rangle Y - \langle X, D_X \nu \rangle \nu \\ &= -\frac{1}{W^{\frac{1}{2}}} \left(hz(X, X) - \frac{sf}{6} \right) Y = \frac{1}{2W^{\frac{1}{2}}} (h\rho + \frac{sf}{3}) Y = bY, \end{aligned}$$

where $b = \frac{1}{2W^{\frac{1}{2}}} (h\rho + \frac{sf}{3})$.

From $\delta r = -\frac{1}{2}ds = 0$, we have

$$D_X r(X, \nu) + D_Y r(Y, \nu) + D_\nu r(\nu, \nu) = 0.$$

From which

$$\begin{aligned} (18) \quad 0 &= X(r(X, \nu)) - r(D_X X, \nu) - r(X, D_X \nu) + Y(r(Y, \nu)) \\ &\quad - r(Y, D_Y \nu) - r(D_Y Y, \nu) + \nu(\rho), \end{aligned}$$

where $D_Y Y = 0$ and $D_\nu \nu = 0$, and the fact that $z(X_i, X_j) = 0$ for orthonormal vector fields X_i with $i \neq j$ (see [7]), we have

$$(19) \quad \nu(\rho) = 0$$

on Σ , which is the first statement of Theorem 1.2. Taking the Lie derivative of (18) with respect to ν gives

$$\begin{aligned} (20) \quad 0 &= \nu X(r(X, \nu)) + \nu Y(r(Y, \nu)) + \nu \nu(\rho) - \nu(r(D_X X, \nu)) \\ &\quad - \nu(r(D_Y Y, \nu)) - \nu(r(X, D_X \nu)) - \nu(r(Y, D_Y \nu)). \end{aligned}$$

The following Contentions from 1 to 6 calculate each terms in (20) to derive the desired equation for $\nu \nu(\rho)$.

Contention 1. On Σ^B

$$(21) \quad \nu(r(\nu, X)) = 0,$$

$$(22) \quad \nu X(r(X, \nu)) = 0.$$

Proof. From $\delta r = 0$, we have

$$(23) \quad D_X r(X, X) + D_Y r(Y, X) + D_\nu r(\nu, X) = 0.$$

The first term of the left-hand side of (23) is

$$(24) \quad D_X r(X, X) = X(r(X, X)) - 2r(D_X X, X) = -\frac{1}{2}X(\rho) - 2br(Y, X) = 0,$$

where we used the fact that $r(X, X) = -\frac{1}{2}\rho + \frac{s}{3}$, $X(\rho) = 0$, and $r(X, Y) = 0$ on Σ . The second term of the left-hand side of (23) is

$$(25) \quad D_Y r(Y, X) = Y(r(Y, X)) - r(Y, D_Y X) = 0,$$

where $\langle D_Y X, Y \rangle = Y\langle X, Y \rangle - \langle X, D_Y Y \rangle = 0$. The last term is

$$(26) \quad D_\nu r(\nu, X) = \nu(r(\nu, X)) - r(\nu, D_\nu X) = \nu(r(\nu, X))$$

by (16). Substitution of (24), (25), and (26) into (23) gives (21), which is the first statement of Contention 1. For the second statement, by (13), (15), and (21) we have $[\nu, X] = 0$, and

$$\nu X(r(X, \nu)) = X\nu(r(X, \nu)) + [\nu, X](r(X, \nu)) = 0.$$

□

Contention 2. On Σ^B we have

$$(27) \quad \nu(r(\nu, Y)) = -\frac{3W^{\frac{1}{2}}\rho}{2h},$$

$$(28) \quad \nu Y(r(Y, \nu)) = -\frac{3}{2}Y(h^{-1}W^{\frac{1}{2}}\rho).$$

Proof. From $\delta r = 0$, we have

$$(29) \quad D_X r(X, Y) + D_Y r(Y, Y) + D_\nu r(\nu, Y) = 0.$$

The first term of the left-hand side of (29) is

$$(30) \quad \begin{aligned} D_X r(X, Y) &= X(r(X, Y)) - r(D_X X, Y) - r(X, D_X Y) \\ &= -br(Y, Y) + br(X, X) = 0, \end{aligned}$$

where $D_X X = bY$ and $D_X Y = \langle D_X Y, X \rangle X = -\langle Y, D_X X \rangle X = -bX$ with $r(X, X) = r(Y, Y)$ on Σ . The second term of the left-hand side of (29) is

$$(31) \quad D_Y r(Y, Y) = Y(r(Y, Y)) = -\frac{1}{2}Y(\rho) = \frac{3W^{\frac{1}{2}}\rho}{2h}$$

by (12). The last term of the left-hand side of (29) is

$$(32) \quad D_\nu r(\nu, Y) = \nu(r(\nu, Y)) - r(\nu, D_\nu Y) = \nu(r(\nu, Y)),$$

where we used the fact that $D_\nu Y = 0$. Substitution of (30), (31), and (32) into (29) gives (27), which is the first statement of Contention 2.

For the second statement, in virtue of (17) and (27) we have

$$\nu Y(r(Y, \nu)) = Y\nu(r(Y, \nu)) + [\nu, Y](r(Y, \nu)) = Y\nu(r(Y, \nu)) = -\frac{3}{2}Y(h^{-1}W^{\frac{1}{2}}\rho),$$

where we used (14) in the third equality. This implies (28). □

Contention 3. On Σ^B we have

$$(33) \quad \nu(r(D_X X, \nu)) = \frac{1}{2}r(\nu, \nu)^2 - \frac{3}{4}\rho^2 - \frac{sf\rho}{4h}.$$

Proof. Let $D_X X = a_1 X + a_2 Y + a_3 \nu$ on a neighborhood of Σ . Note that $a_1 = 0$, $a_3 = 0$, and $a_2 = b$ on Σ^B . Thus, by (19) and (21),

$$\begin{aligned} \nu(r(D_X X, \nu)) &= \nu(a_1 r(X, \nu) + a_2 r(Y, \nu) + a_3 r(\nu, \nu)) \\ &= b\nu(r(Y, \nu)) + \nu(a_3)r(\nu, \nu). \end{aligned}$$

Now, by (13) and (16)

$$\begin{aligned}
 (34) \quad \nu(a_3) &= \nu\langle D_X X, \nu \rangle = \langle D_\nu D_X X, \nu \rangle \\
 &= \langle D_X D_\nu X, \nu \rangle + \langle D_{[\nu, X]} X, \nu \rangle - \langle R(\nu, X) X, \nu \rangle \\
 &= X\langle D_\nu X, \nu \rangle + K(\nu, X) = K(\nu, X).
 \end{aligned}$$

By (27), (34), and $K(X, \nu) = \frac{1}{2}r(\nu, \nu)$, we have (33). $\square$

Contention 4. On Σ^B we have

$$(35) \quad \nu(r(D_Y Y, \nu)) = \frac{1}{2}r(\nu, \nu)^2.$$

Proof. Let $D_Y Y = b_1 X + b_2 Y + b_3 \nu$ on a neighborhood of Σ . Note that, from

$$\begin{aligned}
 D_Y(\Psi^{-\frac{1}{2}}(df)^\top) &= -\frac{1}{2}\Psi^{-\frac{3}{2}}Y(\Psi)(df)^\top + \Phi^{-\frac{1}{2}}D_Y(df)^\top, \\
 \langle D_Y(df)^\top, X \rangle &= \langle D_Y df, X \rangle = 0,
 \end{aligned}$$

$b_1 = \langle D_Y Y, X \rangle = 0$, $b_2 = 0$, and $b_3 = 0$ on Σ^B . Thus

$$\begin{aligned}
 \nu(r(D_Y Y, \nu)) &= \nu(b_1 r(X, \nu) + b_2 r(Y, \nu) + b_3 r(\nu, \nu)) \\
 &= \nu(b_3) r(\nu, \nu).
 \end{aligned}$$

Now, by (14) and (17)

$$\begin{aligned}
 (36) \quad \nu(b_3) &= \nu\langle D_Y Y, \nu \rangle = \langle D_\nu D_Y Y, \nu \rangle \\
 &= \langle D_Y D_\nu Y, \nu \rangle + \langle D_{[\nu, Y]} Y, \nu \rangle - \langle R(\nu, Y) Y, \nu \rangle \\
 &= Y\langle D_\nu Y, \nu \rangle + K(\nu, Y) = K(\nu, Y),
 \end{aligned}$$

where we used the fact that on Σ^B

$$\langle D_\nu Y, \nu \rangle = \Psi^{-\frac{1}{2}}\langle D_\nu(df)^\top, \nu \rangle = 0$$

in the last equality. By (36) and $K(Y, \nu) = \frac{1}{2}r(\nu, \nu)$, we have (35). $\square$

Contention 5. We have

$$(37) \quad \nu(r(X, D_X \nu)) = \frac{1}{4}(r(\nu, \nu) - s)r(\nu, \nu).$$

Proof. Let $D_X \nu = c_1 X + c_2 Y + c_3 \nu$. Note that $c_1 = c_2 = c_3 = 0$ on Σ^B . Then we have

$$(38) \quad \nu(r(X, D_X \nu)) = \nu(c_1)r(X, X) = \nu(c_1)\left(-\frac{1}{2}r(\nu, \nu) + \frac{s}{2}\right).$$

Now we have

$$\begin{aligned}
 (39) \quad \nu(c_1) &= \nu\langle D_X \nu, X \rangle = \langle D_\nu D_X \nu, X \rangle + \langle D_X \nu, D_\nu X \rangle \\
 &= \langle D_X D_\nu \nu, X \rangle + \langle D_{[\nu, X]} \nu, X \rangle - \langle R(\nu, X) \nu, X \rangle \\
 &= -K(\nu, X),
 \end{aligned}$$

where we used (13) and (16) in the third and fourth equality, and $D_\nu \nu = 0$ in the fourth equality. Substitution of (39) into (38) with $K(X, \nu) = \frac{1}{2}r(\nu, \nu)$ gives (37). $\square$

Contention 6. We have

$$(40) \quad \nu(r(Y, D_Y \nu)) = \frac{1}{4}(r(\nu, \nu) - s)r(\nu, \nu).$$

Proof. The proof is similar with that of Contention 4. $\square$

Substituting Contentions from 1 to 6 into (20) gives

$$\nu \nu(\rho) = \frac{3}{2}r(\nu, \nu)^2 - \frac{s}{2}r(\nu, \nu) - \frac{3}{4}\rho^2 - \frac{sf\rho}{4h} + \frac{3}{2}Y(h^{-1}W^{\frac{1}{2}}\rho).$$

From

$$\begin{aligned} Y(W) &= 2\langle D_Y df, df \rangle = -W^{\frac{1}{2}}(h\rho + \frac{sf}{3}), \\ \frac{3}{2}Y(h^{-1}W^{\frac{1}{2}}\rho) &= -\frac{6W\rho}{h^2} - \frac{3}{4}\rho^2 - \frac{sf}{4h}\rho \end{aligned}$$

by (12), we have (10). This completes the proof of Lemma 3.2.

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