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On Coefficient Inequalities of Starlike Functions Related to the q -Analog of Cosine Functions Defined by the Fractional q -Differential Operator

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Abstract: This article extends the study of q -versions of analytic functions by introducing and studying the association of starlike functions with trigonometric cosine functions, both defined in their q -versions. Certain coefficient inequalities like coefficient bounds, Zalcman inequalities, and both Hankel and Toeplitz determinants for the new version of starlike functions are investigated. It is worth mentioning that most of the determined inequalities are sharp with the support of relevant extremal functions.

Keywords: analytic function; q -derivative (or difference) operator; factorial functional; exponential function; univalent function; q -starlike functions

MSC: 30C45; 30C50; 30C80; 11B65; 47B38



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1. Introduction and Preliminaries

Recent mathematical studies have shown that the quantum calculus theory plays a vital role in numerous dimensions in the physical sciences and associated domains. This is true for both fractional and fundamental natures. When we examined the role and significance of differential equations, in this instance the fractional ones, we concluded that they have enlarged their range of engineering applications, particularly when modeling control system issues or problems that arise in control systems. Differential equations may simulate a variety of physical events, although fractional types have a far wider range than conventional ones, e.g., see [1–5]. When the applications are firmly grounded in mathematical premises, an astounding discovery emerges. In recent years, there has been a wonderful synthesis of a more generalized form of classical calculus, namely fractional calculus, with analytic functions. Consequently, many exceptional outcomes in this area have been demonstrated in a short period of time. The fact that this generalized form is far more potent in its spectrum of applications has drawn the attention of pure and applied mathematicians, as well as engineers. The modeling of engineering issues as non-linear differential equations was fairly widespread before the examination of differential equations, including fractional-order derivatives, was even presented, despite the fact that they were rather difficult to manage. However, this new differential operator related to

fractional powers has been shown to be a compelling tool for nonlinear problems; see [6–8]. Many mathematicians have sought to define numerous operators that are connected to analytic function subfamilies and are used to solve differential equations with fractional and algebraic exponents. To gain a mathematical understanding and to provide a solid mathematical foundation for these new emerging concepts, Srivastava provided a comprehensive theory on q -calculus in terms of GFT, a branch of research that deals with complex variables and discusses functions geometrically, in his book [9]. Thus, one may reliably link q -calculus to its fractional analog. Srivastava continued to investigate the essentials of this theory while also exploring new aspects following his outstanding work in this book. He and his colleague Bansal explored a very specific set of functions known as q -Mittag–Leffler functions in terms of any of their elements belonging to close-to-convex functions and demonstrated the requirements. This work is available for interested readers in [10]. There are other works that may be cited as references here, but continuing with the same author, we also discovered another article that defined the associated operators in classical, q -, and fractional q -calculus. He also included the application side of all these specified expressions in GFT. Now, we shall look at the formal definition of the q -derivative, followed by the notion of analytic functions. If the constraints $f(0) = 0, f'(0) = 1$ are satisfied by the analytic function f , which is defined in the open unit disc $\bar{\delta} = \{z \in \mathbb{C} : |z| < 1\}$, then these functions belong to class A , whose Taylor series can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \bar{\delta}. \quad (1)$$

A function f is called univalent in this open unit disk $\bar{\delta}$ if f and $\bar{\delta}(f)$ have a one-to-one relationship, i.e., if $f(z_1) = f(z_2)$ results in $z_1 = z_2$. The constraints $f(0) = 0, f'(0) = 1$ normalize the functions in the class S , which are not only analytic but also univalent in $\bar{\delta}$, indicating that

$$S = \{f \in A : f \text{ is univalent in } \bar{\delta}\}.$$

A basic mathematical characteristic is starlikeness. A straight line connecting each point in a collection to a fixed point creates a domain that resembles a star. If we define this with reference to the fixed point, let us say z_0 , the domain is said to be starlike if all such straight lines lie completely within it. Observing from this fixed point, if each and every point in the domain can be seen, the domain is said to be star-shaped or starlike geometrically. The starlike function is a function that maps $\bar{\delta}$ onto a starlike domain and has the origin as its fixed point. The class S^* of starlike univalent functions is composed of all functions of class S that meet the requirement $z \Re \left(\frac{zf'(z)}{f(z)} \right) > 0, z \in \bar{\delta}$. The next significant class is class P , which is composed of all analytic functions p that are normalized by the requirement $p(0) = 1$ such that $\Re p(z) > 0, z \in \bar{\delta}$. That is, the functions from this class must have their series expression as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} t_n z^n. \quad (2)$$

Let w be an analytic function in $\bar{\delta}$ if $w(0) = 0$ and $|w(z)| < 1$ for $z \in \bar{\delta}$; then, w is the Schwarz function. The existence of Schwarz function w is such that the two analytic functions $f(z)$ and $g(z)$ can be linked as follows:

$$f(z) = g(w(z)), \quad z \in \bar{\delta}, \quad (3)$$

then these functions f and g are related to each other through the subordination relationship, and this can be represented as $f \prec g$. When we use the relationship mentioned in (3), we can write the definitions of the well-known classes P and S^* as follows.

$$P = \left\{ p : p(0) = 1 \text{ and } p(z) \prec \frac{1+z}{1-z}, z \in \bar{\delta} \right\},$$

$$\mathbf{S}^* = \left\{ f \in \mathbf{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in \bar{\delta} \right\}.$$

Now, we give some definitions from quantum calculus, generally known as the q -calculus which is the calculus without the concept of limits. It is used to extend univalent functions theory and to develop various sub-collections of analytic functions. In q -calculus, the following is the q -derivative of a complex-valued function f that is defined in the domain $\mathbf{D}$.

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \quad (4)$$

where $0 < q < 1$. Also,

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

assuming that the limit exists and the derivative $D_q f$ of function f provided in (1) is represented by the Maclaurin series

$$D_q f(z) = \sum_{n=0}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } n \in \mathbb{C}, \\ \sum_{n=0}^{n-1} q^n, & \text{if } n \in \mathbb{N}. \end{cases} \quad (5)$$

Factorial function has q -analog of the form given as;

$$[l]_q! = \prod_{j=1}^l [j]_q = [1]_q [2]_q \cdots [l]_q. \quad (6)$$

Exponential functional has q -analog of the form given as;

$$e_q(z) = \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!}, \quad 0 < q < 1; |z| < \frac{1}{1-q}.$$

In the theory of analytic and univalent functions, the study of coefficients of Maclaurin's series is required. One of the coefficient problems is the Hankel determinant. Hermann Hankel invented the Hankel matrix. It is a matrix with identical elements along every skew diagonal, and those elements are the coefficient of a specific power series of analytic functions. This determinant is a significant figure that helps in describing the characteristics of the associated analytic functions. Pommerenke [11] suggested the following determinant, which is q th Hankel determinant for analytic functions:

$$H_{j,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \end{vmatrix}, \quad j, n \in \mathbb{N}.$$

We are interested in finding the upper bound of this determinant. With certain variations in j and n , we see that the above-defined q th determinant $H_{j,n}(f)$ then reduces to the following determinants. For $j = 2, n = 1$, we have

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2,$$

which is further reduced to the following well-known Fekete–Szegő functional.

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2. \quad (7)$$

For $j = 2, n = 2$, one can have

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \quad (8)$$

For $j = 3, n = 1$, one can have

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= a_5(a_3 - a_2^2) - a_4(a_4 - a_2a_3) + a_3(a_2a_4 - a_3^2). \end{aligned} \quad (9)$$

This implies that

$$|H_{3,1}(f)| \leq |a_5| |a_3 - a_2^2| + |a_4| |a_4 - a_2a_3| + |a_3| |H_{2,2}(f)|. \quad (10)$$

The span of applications of the Hankel determinants has been in various technological studies, particularly those where mathematical tools are used to a large extent. For example, they are used in the theory of Markov processes, and then we see their applications in the solutions of non-stationary signals in the Hamburger moment problem, just to name a few, and readers who are interested in finding the use of Hankel determinants in solutions of the above-stated problems can access [12–14].

Another important type of determinant other than the ones defined by Hankel is the Toeplitz determinant which is defined as follows.

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}; \quad q, n \geq 1. \quad (11)$$

Both the Hankel and Toeplitz determinants are related to each other by having constant entries in their diagonals and the reverse diagonals. The determinants given and named after Toeplitz have these special entries along their diagonals; on the other hand, Hankel has similar entries on its reverse diagonals. The Toeplitz demonstrates a huge range of applications; just to take an idea of how they are used, one can see [15], which gives a clearer view of how this determinant is used in diversity to almost all branches of applicable mathematics, as well as the pure mathematics. If one needs to see the recent research on Hankel determinants, one can find that several authors who have worked to explore the second and third-order Hankel determinants, denoted by $H_{2,2}$ and $H_{3,1}$, respectively, to study a variety of groups of functions. Many research articles can be found to explore similar ideas, like [16–19]. Though the articles [13,20,21] show some very interesting ways in which this determinant has been used and explored, we can find very little exploration done by the researchers to find the actual span of Toeplitz determinants. This is equally true that several advanced research articles comprising applications of these determinants in the literature for a huge range of functions can be found. To find some interesting results, one can read Janteng et al. [22,23] who did marvelous research on the second Hankel determinant, and they did it for such a function that has derivative with positive real part. They also studied it for starlike as well as convex functions. In the literature, we find some other researchers like Bansal [24] and Lee et al. [25], who, in their research, brought the second Hankel determinant for analytic functions under discussion.

Bansal et al. [18] continued working in this direction, followed by Zaprawa [26], as well as Zhang et al. [27] and then Babalola [16], who presented their research of discovering the third-order Hankel determinant, particularly meant for several univalent functions. Raza et al. [28] and Shi et al. [29,30] were inspired by this direction of research and explored the properties of third Hankel determinants, focusing on calculating their upper bounds. They carried this particular research for collections of analytic functions whose geometry is related to the cardioid domain and for Bernoulli's lemniscate. They also presented similar results for an exponential function. The work presented by Mahmood et al. [31] was a continuation of earlier work on the third Hankel determinant, but they did the related results for the subgroup of functions that are q -starlike. The research was then advanced by Zhang et al. [32], who very recently calculated the next-order Hankel determinants and compared them to the ones calculated before. They carried it out for fourth order. They presented their work on determinants of starlike functions that were actually connected to the sine function.

Some recent developments of the current year 2023 regarding the sharp bounds of the third Hankel determinant for certain subclasses of starlike functions are listed in Table 1.

Table 1. Sharp bounds on $|H_{3,1}(f)|$ for some subclasses of $\mathcal{S}^*$.

Author/s	Type of Starlike Functions	Sharp Bound	Reference
Riaz et al.	subordinated by $1 + \frac{4}{5}z + \frac{1}{5}z^4$	1/225	[33]
Riaz and Raza	subordinated by $z + \sqrt{1+z^2}$	1/9	[34]
Wang et al.	subordinated by $1 + \sinh^{-1}(z)$	1/9	[35]
Deniz et al.	subordinated by $\frac{z}{\ln(1+z)}$	43/576	[36]
Marimuthu et al.	subordinated by $\cos z$	139/576	[37]
Li et al.	symmetric connected with exponential function	0.0883	[38]
Tang et al.	symmetric, subordinated by $1 + \frac{4}{5}z + \frac{1}{5}z^4$	0.047	[39]

Since our focus is to give a review of the works related to the Hankel as well as the Toeplitz determinants, we observe that Ali et al. [20] were working on those Toeplitz matrices whose elements are actually the coefficients of different functions, to name a few, e.g., starlike and the close-to-convex functions. There is a remarkable number of research articles coming in this direction. One of these works is the one given by Tang et al. [40] who carried out their research on both the Hankel and Toeplitz determinants, where Hankel was of third order. This work was for a subclass of q -starlike functions, which were multivalent and of order α . Then we see the contribution of Zhang et al. [41], who worked in similar directions with the same type and order of determinants, but the innovation was in the type of function that was considered. Their focus was to prove alike results for starlike functions that are defined by the sine function. The referred work by Ramachandran et al. [42] is a piece of work that shows the derivation of an estimation for these determinants, which have bounded domains by conical sections, and those involve the Ruscheweyh derivative. This brief overview of the work done so far in this direction shows a vital interest of researchers of Hankel determinants who are studying these for a variety of analytic and univalent functions. For $q = 3, n = 1$,

$$T_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_4 & a_1 \end{vmatrix}. \quad (12)$$

For $q = 3, n = 2$,

$$T_{3,2}(f) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}. \quad (13)$$

An interesting result from article [43] was used to find sharp limits for $|H_{3,1}(f)|$. For more in-depth work on Hankel determinants, see [44–49]. In [50], a new form of the fourth Hankel determinant is provided. This form is examined for a new subclass of analytic functions that is introduced, and the fourth-order Hankel determinant's upper bound for this class is found. In [51], a novel category of analytic functions connected to exponential functions is presented, and the upper bound of the third Hankel determinant is formulated. Motivated by the above-mentioned works, we intend to contribute the following to the literature on inequalities related to analytic functions.

- To introduce a novel class $S_{q\cos}^*$ of q -starlike functions which are subordinated by q -cosine function.
- To find the sharp coefficient bounds for functions of class $S_{q\cos}^*$.
- To establish the Zalcman inequalities for functions of class $S_{q\cos}^*$.
- To find the upper bounds of Hankel and Toeplitz determinants for $S_{q\cos}^*$.

Now, to proceed with the above, we define the following.

Definition 1. The q -version of the trigonometric cosine function is given as follows:

$$\cos_q(z) = \frac{e_q(iz) + e_q(-iz)}{2} \quad (14)$$

and the series of (14) is given as

$$\cos_q(z) = \frac{1}{[1]_q!}z - \frac{1}{[3]_q!}z^3 + \frac{1}{[5]_q!}z^5 - \frac{1}{[7]_q!}z^7 + \dots \quad (15)$$

Definition 2. If a function $f \in \mathbf{S}$ fulfills the following criteria, then it is said to belong to the class $S_{q\cos}^*$.

$$\frac{zD_q f(z)}{f(z)} \prec \cos_q(z), \quad z \in \bar{\delta}. \quad (16)$$

That is,

$$S_{q\cos}^* = \left\{ f \in A : \frac{zD_q f(z)}{f(z)} \prec \cos_q(z) \right\}. \quad (17)$$

The class $S_{q\cos}^*$ generalizes the class S_c^* of starlike functions related to the function $\cos(z)$ and $\lim_{q \rightarrow 1^-} S_{q\cos}^* \equiv S_c^*$. The introduction and study of class S_c^* was made by Tang et al. [52]. The following lemmas are required to proceed with our main results.

2. A Set of Lemmas

Lemma 1 ([53]). If $p(z) = 1 + \sum_{n=1}^{\infty} t_n z^n \in \mathbf{P}$, then

$$\begin{aligned} 2t_2 &= t_1^2 + \alpha(4 - t_1^2), \\ 4t_3 &= t_1^3 + 2(4 - t_1^2)t_1\alpha - (4 - t_1^2)t_1\alpha^2 + 2(4 - t_1^2)(1 - |\alpha|^2)\beta, \end{aligned}$$

for some α ($|\alpha| \leq 1$), β ($|\beta| \leq 1$).

Lemma 2 ([54]). Let $p \in \mathbf{P}$ be the function that defines (2). Then,

$$|t_n| \leq 2, \quad (n \in \mathbf{N}). \quad (18)$$

Lemma 3 ([55]). Let $p \in \mathbf{P}$ be the function that defines (2). Then,

$$\left| t_2 - vt_1^2 \right| \leq 2 - v \left| t_1^2 \right| \text{ for } 0 < v \leq \frac{1}{2}. \quad (19)$$

Lemma 4 ([55]). Let $p \in \mathbf{P}$ be the function that defines (2). Then,

$$|vt_n - t_k t_{n-k}| \leq \begin{cases} 2|2 - v|, & v \leq 1, \\ 2v, & v \geq 1. \end{cases} \quad (20)$$

3. Main Results and Their Demonstration

Theorem 1. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$\begin{aligned} |a_3| &\leq \frac{1}{q(1+q)^2}, \\ |a_4| &\leq \frac{4\sqrt{3}}{9q(1+q+q^2)[2]_q}, \\ |a_5| &\leq \frac{2q^6+6q^5+10q^4+12q^3+10q^2+6q+6}{q^2(1+q)^4(1+q^2)^2(1+q+q^2)}. \end{aligned} \quad (21)$$

The first two inequalities show the sharp bounds.

Proof. If $f \in S_{q\cos}^*$, then from the relationships given in (3) and (17), we have

$$\frac{zD_q f(z)}{f(z)} = \cos_q(w(z)), \quad z \in \bar{\delta},$$

where $w(z) = \frac{p(z)-1}{1+p(z)}$. If the function $p(z)$ can be written in the form of (2), then

$$w(z) = \frac{t_1 z + t_2 z^2 + t_3 z^3 + \dots}{2 + t_1 z + t_2 z^2 + t_3 z^3 + \dots}.$$

Now,

$$\begin{aligned} \cos_q(w(z)) &= \cos_q\left(\frac{t_1 z + t_2 z^2 + t_3 z^3 + \dots}{2 + t_1 z + t_2 z^2 + t_3 z^3 + \dots}\right) \\ &= 1 - \frac{t_1^2}{4[2]_q!} z^2 + \frac{t_1(-2t_2 + t_1^2)}{4[2]_q!} z^3 \\ &\quad + \frac{(-8t_1 t_3 + 12t_1^2 t_2 - 3t_1^4 - 4t_2^2)[4]_q! + t_1^4 [2]_q!}{16[2]_q! [4]_q!} z^4 + \dots \end{aligned} \quad (22)$$

Consider

$$\begin{aligned} \frac{zD_q f(z)}{f(z)} &= \frac{z\{(z+a_2 z^2+a_3 z^3+\dots)-(qz+qa_2 z^2+qa_3 z^3+\dots)\}}{z(1-q)(z+a_2 z^2+a_3 z^3+\dots)} \\ &= 1 + a_2 q z + \{a_3 q(1+q)a_3 - a_2^2 q\} z^2 + \{q(1+q+q^2)a_4 - (q^2+2q)a_2 a_3 + qa_2^3\} z^3 \\ &\quad + q\{a_5 q^3 + (-a_2 a_4 + a_5)q^2 + (-a_3^2 + a_3 a_2^2 - a_2 a_4 + a_5)q \\ &\quad + 3a_3 a_2^2 - 2a_2 a_4 - a_2^4 - a_3^2 + a_5\} z^4 + \dots \end{aligned} \quad (23)$$

The comparison of coefficients of z , z^2 , z^3 and z^4 , along with precise computation leads us to

$$a_2 = 0, \quad (24)$$

$$a_3 = -\frac{t_1^2}{4[2]_q! q(1+q)}, \quad (25)$$

$$a_4 = -\frac{t_1(2t_2 - t_1^2)}{4[2]_q! q(1+q+q^2)}, \quad (26)$$

$$a_5 = \frac{1}{16q^2(q^4+2q^3+2q^2+2q+1)[2]_q!^2[4]_q!} \left(([2]_q!q^2([2]_q!-3[4]_q!) + [2]_q!q([2]_q!-3[4]_q!) + [4]_q!)t_1^4 + 12[2]_q![4]_q!q(1+q)t_1^2t_2 - 8[2]_q![4]_q!q(1+q)t_1t_3 - 4[2]_q![4]_q!q(1+q)t_2^2 \right). \quad (27)$$

Applying Lemma 2 to (25), we obtain $|a_3| \leq \frac{1}{q(1+q)^2}$. For sharpness, consider the function

$$\frac{zD_q f_0(z)}{f_0(z)} = \cos_q(z) = \frac{1}{[1]_q!}z - \frac{1}{[3]_q!}z^3 + \frac{1}{[5]_q!}z^5 - \frac{1}{[7]_q!}z^7 + \dots \quad (28)$$

Now using (23), (28) and comparing them, we obtain

$$f_0(z) = z - \frac{1}{q(1+q)^2}z^3 + \dots \quad (29)$$

Now, consider

$$\begin{aligned} |a_4| &= \left| -\frac{t_1(2t_2 - t_1^2)}{4[2]_q!q(1+q+q^2)} \right| \\ &= \frac{1}{2[2]_q!q(1+q+q^2)} |t_1| \left| t_2 - \frac{1}{2}t_1^2 \right|. \end{aligned}$$

Applying Lemma 3 with $\nu = \frac{1}{2}$, we obtain

$$|a_4| \leq \frac{1}{2[2]_q!q(1+q+q^2)} |t_1| \left(2 - \frac{1}{2}|t_1^2| \right).$$

Letting $|t_1| = t$, we obtain

$$\begin{aligned} |a_4| &\leq \frac{1}{2[2]_q!q(1+q+q^2)} t \left(2 - \frac{1}{2}t^2 \right) \\ &= \omega(t). \end{aligned}$$

The function $\omega(t)$ has maximum value at $t = \frac{2}{\sqrt{3}}$ which gives that $\omega(t) \leq \frac{4\sqrt{3}}{9q(1+q+q^2)[2]_q}$.

For the sharpness, consider $t_0 = \frac{2}{\sqrt{3}}$ and $g_0(z) = \frac{1-z^2}{1-t_0z+z^2}$ such that

$$w_0(z) = \frac{g_0(z) - 1}{g_0(z) + 1} = \frac{z(3z - \sqrt{3})}{-3 + \sqrt{3}z}.$$

It is easy to see that $w_0(0) = 0$ and $|w_0(z)| < 1$. Hence the function

$$\frac{zD_q f_1(z)}{f_1(z)} = \cos_q(w_0(z)) = 1 - \frac{z^2}{3([2]_q)!} + \frac{4\sqrt{3}z^3}{9([2]_q)!} + \frac{z^4}{9([4]_q)!} + \dots \quad (30)$$

is in class $S_{q\cos}^*$. Now, using (23) and (30) and comparing, we obtain

$$f_1(z) = z - \frac{1}{3q(q+1)[2]_q}z^3 + \frac{4\sqrt{3}}{9q(1+q+q^2)[2]_q}z^4 + \dots \quad (31)$$

Hence $a_4 = \frac{4\sqrt{3}}{9q(1+q+q^2)[2]_q}$. Now, applying Lemma 1 in (27), we obtain

$$|a_5| = \left| \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\left([2]_q!q^2([2]_q! - 3[4]_q!) \right. \right. \right. \\ \left. \left. \left. [2]_q!q \left(([2]_q! - 3[4]_q!) + [4]_q! \right) \right) t_1^4 + 6[2]_q![4]_q!q(1+q) \right. \right. \\ \left. \left. \left(t_1^2 + \alpha(4 - t_1^2) \right) t_1^2 - 8[2]_q![4]_q!q(1+q) \left(t_1^3 + 2(4 - t_1^2)t_1\alpha - \right. \right. \right. \\ \left. \left. \left. (4 - t_1^2)t_1\alpha^2 + 2(4 - t_1^2)(1 - |\alpha|^2)\beta \right) t_1 - 4[2]_q![4]_q!q(1+q) \right. \right. \\ \left. \left. \left. \left(t_1^2 + \alpha(4 - t_1^2) \right)^2 \right) \right) \right|.$$

This implies that

$$|a_5| = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\left(2[2]_q!(q^2 + q) + [4]_q! \right) t^4 \right. \\ \left. + 2[2]_q![4]_q!q(1+q)(4 - t^2)t^2\alpha^2 - 4[2]_q!q(1+q)(4 - t^2)^2\alpha^2 \right. \\ \left. - 4[2]_q![4]_q!(q^2 + q)t(4 - t^2)(1 - |\alpha|^2)\beta \right).$$

Using triangular inequality and letting $t_1 = t, |\alpha| = h$, we obtain

$$|a_5| \leq \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\left(2(q^2 + q)[2]_q! + [4]_q! \right) t^4 \right. \\ \left. + 2[2]_q![4]_q!(q^2 + q)(4 - t^2)t^2h^2 + 4[2]_q!(q^2 + q)(4 - t^2)^2h^2 \right. \\ \left. + 8[2]_q![4]_q!(q^2 + q)(4 - t^2) \right).$$

We assume that

$$\Psi_q(t, h) = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\left(2[2]_q!(q^2 + q) + [4]_q! \right) t^4 \right. \\ \left. + 2[2]_q![4]_q!(q^2 + q)(4 - t^2)t^2h^2 + 4[2]_q!(q^2 + q)(4 - t^2)^2h^2 \right. \\ \left. + 8[2]_q![4]_q!(q^2 + q)(4 - t^2) \right).$$

Upon partial differentiation, we obtain

$$\frac{\partial \Psi_q}{\partial h} = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(4[2]_q![4]_q!(q^2 + q)(4 - t^2) \right. \\ \left. t^2t + 8[2]_q!(q^2 + q)(4 - t^2)^2h \right) > 0.$$

It implies $\Psi_q(t, h)$ is an increasing function in $[0, 1]$. Therefore,

$$\max(\Psi_q(t, h)) = \Psi_q(t, 1) \\ = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\begin{aligned} & \left(2[2]_q!(q^2 + q) + [4]_q! \right) t^4 \\ & + 2[2]_q![4]_q!(q^2 + q)(4 - t^2)t^2 \\ & + 4[2]_q!(q^2 + q)(4 - t^2)^2 + \\ & + 8[2]_q![4]_q!(q^2 + q)(4 - t^2) \end{aligned} \right).$$

Set

$$\Phi_q(t) = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\begin{aligned} & (2[2]_q!(q^2 + q) + [4]_q!)t^4 \\ & + 2[2]_q![4]_q!(q^2 + q)(4 - t^2)t^2 \\ & + 4[2]_q!(q^2 + q)(4 - t^2)^2 + \\ & + 8[2]_q![4]_q!(q^2 + q)(4 - t^2) \end{aligned} \right),$$

which gives

$$\Phi'_q(t) = \frac{1}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} (Lt^3 + Mt),$$

where

$$\begin{aligned} L &= -8q^9 - 40q^8 - 96q^7 - 148q^6 - 164q^5 - 132q^4 - 48q^3 + 28q^2 + 28q + 4, \\ M &= -64q^3 - 128q^2 - 64q. \end{aligned}$$

If we set $\Phi'_q(t) = 0$, then we obtain three roots, which are $t_1 = 0$,

$$\begin{aligned} t_2 &= \frac{4\sqrt{(-2q^7 - 6q^6 - 10q^5 - 11q^4 - 9q^3 - 4q^2 + 5q + 1)q}}{2q^7 - 6q^6 + 10q^5 + 11q^4 + 9q^3 + 4q^2 - 5q - 1}, \\ t_3 &= -\frac{4\sqrt{(-2q^7 - 6q^6 - 10q^5 - 11q^4 - 9q^3 - 4q^2 + 5q + 1)q}}{2q^7 - 6q^6 + 10q^5 + 11q^4 + 9q^3 + 4q^2 - 5q - 1}. \end{aligned}$$

Calculations give that

$$\Phi''_q(t_1) < 0,$$

which shows that the function $\Phi_q(t)$ has maximum value at $t_1 = 0$. So

$$\begin{aligned} |a_5| &\leq \Phi_q(0) = \frac{2(1+q)(2 + [4]_q!)}{q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \\ &= \frac{2q^6 + 6q^5 + 10q^4 + 12q^3 + 10q^2 + 6q + 6}{q^2(1+q)^4(1+q^2)^2(1+q+q^2)}. \end{aligned}$$

□

Taking $q \rightarrow 1^-$ in the previous expression, we conclude that the following bound is a much more improved bound than the one proved in [37].

Corollary 1. *If the series form of $f \in S_c^*$ is specified in (1), then*

$$|a_3| \leq \frac{1}{4}, |a_4| \leq \frac{2}{9\sqrt{3}}, |a_5| \leq \frac{52}{192}.$$

4. Zalcman and Generalized Zalcman Conjecture

A noteworthy hypothesis for univalent functions was put forth by Zalcman in 1960, and Ma [56] provided its generalized form in 1999. These hypotheses have been proven for several univalent function subclasses. Zalcman's hypothesis states that every $f \in \mathbf{S}$ with the form (1) fulfills the following sharp inequality.

$$|a_n^2 - a_{2n-1}| \leq (1-n)^2, \quad n \geq 2. \quad (32)$$

The generalized Zalcman's hypothesis claims that the following inequality is satisfied by $f \in \mathbf{S}$ having Taylor series of the form (1).

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), \quad \forall m, n \in \mathbb{N}, n \geq 2, m \geq 2. \quad (33)$$

After assigning specific values to n and m , we are looking forward to finding the upper bounds for these inequalities, which is meant to be true for the considered class $S_{q\cos}^*$. The inequality (32) has the following form for $n = 2$.

$$|a_2^2 - a_3| \leq 1.$$

Theorem 2. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$|a_3 - a_2^2| = |a_3| \leq \frac{1}{q(1+q)^2}. \quad (34)$$

This is sharp for the extremal function f_0 defined in (29).

Proof follows easily by Theorem 1. This result is reduced to the following by allowing $q \rightarrow 1^-$.

Corollary 2. If the series form of $f \in S_c^*$ is specified in (1), then,

$$|a_3 - a_2^2| \leq \frac{1}{4}.$$

The inequality (33) is reduced to $|a_2 a_3 - a_4| \leq 2$ for $n = 3, m = 2$. We go through it below:

Theorem 3. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then,

$$|a_4 - a_3 a_2| = |a_4| \leq \frac{4\sqrt{3}}{9q(1+q+q^2)[2]_q}. \quad (35)$$

This is sharp for the extremal function f_1 defined in (31).

Proof follows easily by Theorem 1. This result is reduced to the following one by allowing $q \rightarrow 1^-$.

Corollary 3. If the series form of $f \in S_c^*$ is specified in (1), then,

$$|a_4 - a_3 a_2| \leq \frac{2}{9\sqrt{3}}.$$

The inequality (33) is reduced to $|a_5 - a_3^2| \leq 4$ for $n = 3, m = 3$. We go through it below:

Theorem 4. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then,

$$|a_5 - a_3^2| \leq 4. \quad (36)$$

Proof. From (25) and (27), consider

$$a_5 - a_3^2 = \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(- (3q^8 + 16q^7 + 39q^6 + 62q^5 + 72q^4 + 61q^3 + 36q^2 + 13q + 2)t_1^4 + 12[2]_q![4]_q!(1+q)t_1^2t_2 - 8[2]_q![4]_q!(1+q)t_1t_3 - 4[2]_q![4]_q!(1+q)t_2^2 \right).$$

Using Lemma 1 and, after simplification, we obtain

$$a_5 - a_3^2 = \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\begin{array}{l} -(-q^7 - 3q^6 - 5q^5 - 6q^4 - 4q^3 + 2q + 1)t_1^4 + \\ 2[2]_q![4]_q!(q+1)(4-t_1^2)t_1^2\alpha^2 \\ -[2]_q![4]_q!(q+1)(4-t_1^2)^2\alpha^2 \\ 4[2]_q![4]_q!(q+1)t_1(4-t_1^2)(1-|\alpha|^2)z \end{array} \right).$$

Assuming $|\alpha| = h \in [0, 1]$, $t_1 = t \in [0, 2]$ and using triangular inequality, we have

$$\begin{aligned} & |a_5 - a_3^2| \\ & \leq \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\begin{array}{l} (-q^7 - 3q^6 - 5q^5 - 6q^4 - 4q^3 + 2q + 1)t^4 + \\ 2[2]_q![4]_q!(q+1)(4-t^2)t^2h^2 \\ + [2]_q![4]_q!(q+1)(4-t^2)^2h^2 + \\ 4[2]_q![4]_q!(q+1)t(4-t^2) \end{array} \right) \\ & : = Q_q(t, h). \end{aligned}$$

Differentiating with respect to h gives

$$\frac{\partial Q_q}{\partial h} = \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} \left(\begin{array}{l} 4[2]_q![4]_q!(q+1)(4-t^2)t^2h \\ + 2[2]_q![4]_q!(q+1)(4-t^2)^2h \end{array} \right).$$

Clearly $\frac{\partial Q_q}{\partial h} > 0$, $q \in (0, 1)$, $t \in (0, 2)$ which means that $Q_q(t, h)$ is increasing function for $h \in [0, 1]$. That is

$$\max Q_q(t, h) = Q_q(t, 1) = \frac{\left(\begin{array}{l} (-q^7 - 3q^6 - 5q^5 - 6q^4 - 4q^3 + 2q + 1)t^4 + \\ 2[2]_q![4]_q!(q+1)(4-t^2)t^2 \\ + [2]_q![4]_q!(q+1)(4-t^2)^2 + \\ 4[2]_q![4]_q!(q+1)t(4-t^2) \end{array} \right)}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!}.$$

After simplifications,

$$\Phi_q(t) = \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} (At^4 + Bt^3 + Ct^2 + Dt + E),$$

where

$$\begin{aligned} A &= -q^7 - 3q^6 - 5q^5 - 6q^4 - 4q^3 + 2q + 1, \\ B &= -(4q^8 + 20q^7 + 48q^6 + 76q^5 + 88q^4 + 76q^3 + 48q^2 + 20q + 4), \\ C &= -(8q^8 + 40q^7 + 96q^6 + 152q^5 + 176q^4 + 152q^3 + 96q^2 + 40q + 8), \\ D &= 16q^8 + 80q^7 + 192q^6 + 304q^5 + 352q^4 + 304q^3 + 192q^2 + 80q + 16, \\ E &= 32q^8 + 160q^7 + 384q^6 + 608q^5 + 704q^4 + 608q^3 + 384q^2 + 160q + 32. \end{aligned}$$

Differentiating $\Phi_q(t)$ with respect to t , we obtain

$$\Phi_q'(t) = \frac{1}{16q(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2[4]_q!} (A't^3 + B't^2 + C't + D)$$

with

$$\begin{aligned} A' &= 4(-q^7 - 3q^6 - 5q^5 - 6q^4 - 4q^3 + 2q + 1), \\ B' &= -3(4q^8 + 20q^7 + 48q^6 + 76q^5 + 88q^4 + 76q^3 + 48q^2 + 20q + 4), \\ t' &= -2(8q^8 + 40q^7 + 96q^6 + 152q^5 + 176q^4 + 152q^3 + 96q^2 + 40q + 8), \\ D &= 16q^8 + 80q^7 + 192q^6 + 304q^5 + 352q^4 + 304q^3 + 192q^2 + 80q + 16. \end{aligned}$$

Certain calculations show that $A't^3 + B't^2 + C't + D > 0$ for $t \in (0, 0.6]$ and $q \in (0, 1)$. Also, $A't^3 + B't^2 + C't + D \leq 0$ for $t \in (0.6, 2)$ and $q \in (0, 1)$. This means that $\Phi_q'(t) > 0$ $t \in (0, 0.6]$ and $\Phi_q'(t) \leq 0$ for $t \in (0.6, 2)$. It implies $\Phi_q(t)$ is increasing in $(0, 0.6]$ and $\Phi_q(t)$ is decreasing in $(0.6, 2)$. This means that

$$\Phi_q(t) \leq \Phi_q(0.6) = \frac{\begin{pmatrix} 37.856q^8 + 189.1504q^7 + 453.8832q^6 + \\ 718.6160q^5 + 832.0544q^4 + 718.7456q^3 \\ + 454.272q^2 + 189.5392q + 37.9856 \end{pmatrix}}{(q^4 + 2q^3 + 2q^2 + 2q + 1)} < 4, q \in (0.31, 1).$$

Consequently, we obtain

$$|a_5 - a_3^2| \leq 4.$$

□

5. Hankel Determinants

Theorem 5. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$|H_{2,1}(f)| \leq \frac{1}{q(q+1)^2}.$$

This is sharp for the extremal function f_0 defined in (29).

The proof of above inequality is followed by using (24) and (25) in (7) and then applying Theorem 1.

This result is reduced to the following one proved in [37] by allowing $q \rightarrow 1^-$.

Corollary 4. If the series form of $f \in S_c^*$ is specified in (1), then

$$|H_{2,1}(f)| \leq \frac{1}{4}.$$

Theorem 6. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$|H_{2,2}(f)| \leq \frac{1}{q(q+1)^2}.$$

This is sharp for the extremal function f_0 defined in (29).

The proof of above inequality is followed by using (24) and (25) in (8) and then applying Theorem 1.

This result is reduced to the following one proved in [37] by allowing $q \rightarrow 1^-$.

Corollary 5. If the series form of $f \in S_c^*$ is specified in (1), then

$$|H_{2,2}(f)| \leq \frac{1}{16}.$$

Theorem 7. If the series form of $f \in S_{q \cos}^*$ is specified in (1), then

$$|H_{3,1}(f)| \leq \frac{8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9}{q^2(1+q)^3(1+q+q^2)^2(1+q+q^2+q^3)(q^4+2q^3+2q^2+2q+1)}.$$

Proof. Using (24) in (9), we obtain

$$H_{3,1}(f) = a_5 a_3 - a_4^2 - a_3^3. \quad (37)$$

Now using (25)–(27), we obtain

$$\begin{aligned} H_{3,1}(f) &= \frac{7q^8 + 29q^7 + 61q^6 + 94q^5 + 108q^4 + 93q^3 + 59q^2 + 27q + 6}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} t_1^6 \\ &\quad - \frac{7q^4 + 14q^3 + 17q^2 + 14q + 7}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} t_1^4 t_2 \\ &\quad + \frac{t_1^3 t_3}{8q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\ &\quad - \frac{3q^4 + 6q^3 + 5q^2 + 6q + 3}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} t_1^2 t_2^2. \end{aligned}$$

Applying Lemma 1, we obtain

$$\begin{aligned} H_{3,1}(f) &= \frac{(7q^8 + 29q^7 + 61q^6 + 94q^5 + 108q^4 + 93q^3 + 59q^2 + 27q + 6)}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} t_1^6 \\ &\quad - \frac{(7q^4 + 14q^3 + 17q^2 + 14q + 7)t_1^4(t_1^2 + \alpha(4 - t_1^2))}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\ &\quad + \frac{t_1^3(t_1^3 + 2(4 - t_1^2)t_1\alpha - (4 - t_1^2)t_1^2\alpha + 2(4 - t_1^2)(1 - |\alpha|^2)\beta)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\ &\quad - \frac{(3q^4 + 6q^3 + 5q^2 + 6q + 3)t_1^2(t_1^2 + \alpha(4 - t_1^2))^2}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2}. \\ |H_{3,1}(f)| &= \left| \frac{(8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9)}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} t_1^6 \right. \\ &\quad + \frac{(9q^4 + 18q^3 + 19q^2 + 18q + 9)t_1^4\alpha(4 - t_1^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\ &\quad + \frac{t_1^4\alpha^2(4 - t_1^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\ &\quad - \frac{(3q^4 + 6q^3 + 5q^2 + 6q + 3)t_1^2\alpha^2(4 - t_1^2)^2}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\ &\quad \left. - \frac{t_1^3(4 - t_1^2)(1 - |\alpha|^2)\beta}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \right|. \end{aligned}$$

Using triangular inequality and letting $t_1 = t$, $|\alpha| = h$, we obtain

$$\begin{aligned}
 |H_{3,1}(f)| &= \left(\frac{(8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9)}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} t^6 \right. \\
 &\quad + \frac{(9q^4 + 18q^3 + 19q^2 + 18q + 9)t^4 h(4 - t^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\
 &\quad + \frac{t^4 h^2(4 - t^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\
 &\quad + \frac{(3q^4 + 6q^3 + 5q^2 + 6q + 3)t^2 h^2(4 - t^2)^2}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\
 &\quad \left. + \frac{t^3(4 - t^2)}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \right) := \Omega_q(t, h).
 \end{aligned}$$

Partially differentiating $\Omega_q(t, h)$ regarding “h”, we obtain

$$\begin{aligned}
 \frac{\partial \Omega_q}{\partial h} &= \left(\frac{(9q^4 + 18q^3 + 19q^2 + 18q + 9)t^4(4 - t^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \right. \\
 &\quad + \frac{ht^4(4 - t^2)}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\
 &\quad \left. + \frac{2t(3q^4 + 6q^3 + 5q^2 + 6q + 3)t^2(4 - t^2)^2}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \right).
 \end{aligned}$$

Clearly $\frac{\partial \Omega_q}{\partial h} > 0$ for $q \in (0, 1)$, $t \in (0, 2)$. Hence $\Omega_q(t, h)$ is increasing function in $[0, 1]$ and has its maximum value at $h = 1$, i.e.,

$$\begin{aligned}
 \max(\Omega_q(t, h)) &= \Omega_q(t, 1) \\
 &= \left(\frac{(8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9)}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} t^6 \right. \\
 &\quad + \frac{(9q^4 + 18q^3 + 19q^2 + 18q + 9)t^4(4 - t^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\
 &\quad + \frac{t^4(4 - t^2)}{32q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \\
 &\quad + \frac{(3q^4 + 6q^3 + 5q^2 + 6q + 3)t^2(4 - t^2)^2}{64q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)^2[2]_q!^2} \\
 &\quad \left. + \frac{t^3(4 - t^2)}{16q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)[2]_q!^2} \right) := g_q(t).
 \end{aligned}$$

This implies that

$$g'_q(t) = At^5 + Bt^4 + Ct^3 + Dt^2 + Et,$$

where

$$\begin{aligned} A &= -(27q^8 + 111q^7 + 237q^6 + 366q^5 + 420q^4 + 363q^3 + 231q^2 + 108q + 24) < 0, \\ B &= -(10q^8 + 40q^7 + 90q^6 + 140q^5 + 160q^4 + 140q^3 + 90q^2 + 40q + 10) < 0, \\ C &= 112q^8 + 448q^7 + 944q^6 + 1440q^5 + 1664q^4 + 1440q^3 + 944q^2 + 448q + 112, \\ D &= 24q^8 + 96q^7 + 216q^6 + 336q^5 + 384q^4 + 336q^3 + 36q^2 + 216q + 24, \\ E &= 48q^8 + 192q^7 + 368q^6 + 544q^5 + 640q^4 + 554q^3 + 544q^2 + 192q + 48. \end{aligned}$$

We see that $g'_q(t) > 0$ which implies $g_q(t)$ is an increasing function in $(0, 2)$. So

$$\begin{aligned} g_q(t) &\leq g_q(2) = \frac{8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9}{q^2(q^4 + 2q^3 + 2q^2 + 2q + 1)(1 + q + q^2)[2]_q!^2[4]_q!} \\ &= \frac{8q^8 + 31q^7 + 62q^6 + 92q^5 + 108q^4 + 93q^3 + 64q^2 + 32q + 9}{q^2(1 + q)^3(1 + q + q^2)^2(1 + q + q^2 + q^3)(q^4 + 2q^3 + 2q^2 + 2q + 1)}. \end{aligned}$$

□

This result is reduced to the following improved result that is proved in [37] by allowing $q \rightarrow 1^-$.

Corollary 6. If the series form of $f \in S_c^*$ is specified in (1), then

$$|H_{3,1}(f)| \leq \frac{499}{2304}.$$

6. Toeplitz Determinant

Theorem 8. If the series form of $f \in S_{\cos}^*$ is specified in (1), then

$$|T_{3,1}(f)| \leq 1 + \frac{1}{q^2(1 + q)^4}.$$

This is sharp for the extremal function f_0 defined in (29).

Proof. Using (24) in (12), we have

$$T_{3,1}(f) = \begin{vmatrix} 1 & 0 & a_3 \\ 0 & 1 & 0 \\ a_3 & 0 & 1 \end{vmatrix} = 1 - a_3^2,$$

and

$$|T_{3,2}(f)| \leq 1 + |a_3|^2.$$

Using Theorem 1, we have

$$|T_{3,1}(f)| \leq 1 + \frac{1}{q^2(1 + q)^4}$$

□

This result is reduced to the following by allowing $q \rightarrow 1^-$.

Corollary 7. If the series form of $f \in S_c^*$ is specified in (1), then

$$|T_{3,1}(f)| \leq \frac{17}{16}.$$

Theorem 9. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$|T_{3,2}(f)| \leq \frac{4}{q^3[2]_q!(1+q)^4(1+q+q^2)}.$$

Proof. Using (24) in (13) we have

$$T_{3,2}(f) = \begin{vmatrix} 0 & a_3 & a_4 \\ a_3 & 0 & a_3 \\ a_4 & 0 & 0 \end{vmatrix} = 2a_3^2a_4.$$

and

$$|T_{3,2}(f)| = 2|a_3|^2|a_4|.$$

Using Theorem 1, we have

$$|T_{3,2}(f)| \leq \frac{4}{q^3[2]_q!(1+q)^4(1+q+q^2)}.$$

□

This result is reduced to the following one by allowing $q \rightarrow 1^-$.

Corollary 8. If the series form of $f \in S_c^*$ is specified in (1), then

$$|T_{3,2}(f)| \leq \frac{1}{24}.$$

Theorem 10. If the series form of $f \in S_{q\cos}^*$ is specified in (1), then

$$|T_{4,1}(f)| \leq \frac{\left(\begin{array}{l} q^{16} + 10q^{15} + 47q^{14} + 138q^{13} + 283q^{12} + \\ 428q^{11} + 492q^{10} + 440q^9 + 321q^8 + 222q^7 + \\ 179q^6 + 150q^5 + 96q^4 + 38q^3 + 13q^2 + 10q + 5 \end{array} \right)}{q^4(1+q)^8(1+q+q^2)^2}.$$

Proof. For $q = 4, n = 1$ in (11), we obtain

$$T_{4,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & 1 & a_4 & a_3 \\ a_3 & a_4 & 1 & a_2 \\ a_4 & a_3 & a_2 & 1 \end{vmatrix}.$$

Using (24) and expanding, we obtain

$$|T_{4,1}(f)| \leq \left| 1 - a_4^2 - a_3^2 \right| + \left| a_3^2 \right| \left| a_3^2 - a_4^2 - 1 \right| + \left| a_4^2 \right| \left| 1 + a_3^2 - a_4^2 \right|. \quad (38)$$

Now consider

$$\left| 1 - a_4^2 - a_3^2 \right| = \left| 1 - \frac{4t_1^2t_2^2 + t_1^6 - 4t_1^4t_2}{16[2]_q!^2q^2(q^2 + q + 1)^2} - \frac{t_1^4}{16[2]_q!^2q^2(q + 1)^2} \right|.$$

Using Lemma 1 and taking $t_1 = t \in [0, 2]$ and after simplifications, we obtain

$$\left| 1 - a_4^2 - a_3^2 \right| = \left| 1 - \frac{(4 - t^2)^2 t^2 \alpha^2}{16[2]_q!^2q^2(q^2 + q + 1)^2} - \frac{t^4}{16[2]_q!^2q^2(q + 1)^2} \right|.$$

Now, applying triangle inequality and by replacing $|\alpha| = h$, we obtain

$$\begin{aligned} |1 - a_4^2 - a_3^2| &\leq 1 + \frac{(4-t^2)^2 t^2 h^2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} + \frac{t^4}{16[2]_q!^2 q^2 (q+1)^2} \\ &:= H_q(t, h). \end{aligned} \quad (39)$$

Now consider

$$|a_3^2 - a_4^2 - 1| = \left| \frac{t_1^4}{16[2]_q!^2 q^2 (q+1)^2} - \frac{4t_1^2 t_2^2 + t_1^6 - 4t_1^4 t_2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} - 1 \right|.$$

Using Lemma 1 and let $t_1 = t \in [0, 2]$ and after simplifications, we obtain

$$|a_3^2 - a_4^2 - 1| = \left| \frac{t^4}{16[2]_q!^2 q^2 (q+1)^2} - \frac{(4-t^2)^2 t^2 \alpha^2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} - 1 \right|.$$

Now, applying triangle inequality and by replacing $|\alpha| = h$, we obtain

$$\begin{aligned} |a_3^2 - a_4^2 - 1| &\leq \frac{t^4}{16[2]_q!^2 q^2 (q+1)^2} + \frac{(4-t^2)^2 t^2 h^2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} + 1 \\ &:= H_q(t, h). \end{aligned} \quad (40)$$

Now consider

$$|1 + a_3^2 - a_4^2| = \left| 1 + \frac{t_1^4}{16[2]_q!^2 q^2 (q+1)^2} - \frac{4t_1^2 t_2^2 + t_1^6 - 4t_1^4 t_2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} \right|.$$

Using Lemma 1 and letting $t_1 = t \in [0, 2]$ and after simplifications, we obtain

$$|1 + a_3^2 - a_4^2| \leq \left| 1 + \frac{t^4}{16[2]_q!^2 q^2 (1+q)^2} - \frac{(4-t^2)^2 t^2 \alpha^2}{16[2]_q!^2 q^2 (1+q+q^2)^2} \right|.$$

Now, applying triangle inequality and by replacing $|\alpha| = h$, we obtain

$$\begin{aligned} |1 + a_3^2 - a_4^2| &\leq 1 + \frac{t^4}{16[2]_q!^2 q^2 (q+1)^2} + \frac{(4-t^2)^2 t^2 h^2}{16[2]_q!^2 q^2 (1+q+q^2)^2} \\ &:= H_q(t, h). \end{aligned} \quad (41)$$

From (39)–(41), we have

$$|1 - a_4^2 - a_3^2| = |a_3^2 - a_4^2 - 1| = |1 + a_3^2 - a_4^2| \leq H_q(t, h). \quad (42)$$

Partially differentiating $H_q(t, h)$ regarding “ h ”, we obtain

$$\frac{\partial H_q}{\partial h} = \frac{(4-t^2)^2 t^2 h}{8[2]_q!^2 q^2 (1+q+q^2)^2}.$$

Clearly $\frac{\partial H_q}{\partial h} > 0$ for $q \in (0, 1)$, $t \in (0, 2)$. Hence $H_q(t, h)$ is increasing function in $[0, 1]$ and has its maximum value at $h = 1$.

$$\begin{aligned} \max H_q(t, h) &= H_q(t, 1) = 1 + \frac{(4-t^2)^2 t^2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} + \frac{t^4}{16[2]_q!^2 q^2 (q+1)^2} \\ &:= h_q(t). \end{aligned}$$

This implies that

$$\begin{aligned} h'_q(t) &= \frac{\frac{d}{dt} \left((4-t^2)^2 t^2 \right)}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} + \frac{\frac{d}{dt} t^4}{16[2]_q!^2 q^2 (q + 1)^2} \\ &= \frac{(4-t^2)^2 \cdot 2t + 2(4-t^2)(-2t)t^2}{16[2]_q!^2 q^2 (q^2 + q + 1)^2} + \frac{4t^3}{16[2]_q!^2 q^2 (q + 1)^2} \\ &= \frac{(4-t^2)^2 t}{8[2]_q!^2 q^2 (q^2 + q + 1)^2} - \frac{(4-t^2)t^3}{4[2]_q!^2 q (q^2 + q + 1)^2} + \frac{t^3}{4[2]_q!^2 q^2 (q + 1)^2}. \end{aligned}$$

We see that $h'_q(t) > 0$ for $q \in (0, 1)$ and $t \in (0, 2)$ which implies $h_q(t)$ is an increasing function in $(0, 2)$. So

$$H_q(t, h) \leq h_q(2) = 1 + \frac{1}{[2]_q!^2 q^2 (q + 1)^2}. \quad (43)$$

Using (43) in (42), one may have

$$\left| 1 - a_4^2 - a_3^2 \right| = \left| a_3^2 - a_4^2 - 1 \right| = \left| 1 + a_3^2 - a_4^2 \right| \leq 1 + \frac{1}{[2]_q!^2 q^2 (q + 1)^2}. \quad (44)$$

Using Theorem 1 and (44) in (38), we have

$$|T_{4,1}(f)| \leq \frac{\begin{pmatrix} q^{16} + 10q^{15} + 47q^{14} + 138q^{13} + 283q^{12} + \\ 428q^{11} + 492q^{10} + 440q^9 + 321q^8 + 222q^7 + \\ 179q^6 + 150q^5 + 96q^4 + 38q^3 + 13q^2 + 10q + 5 \end{pmatrix}}{q^4(1+q)^8(1+q+q^2)^2}.$$

□

This result is reduced to the following one by allowing $q \rightarrow 1^-$.

Corollary 9. *If the series form of $f \in S_c^*$ is specified in (1), then*

$$|T_{4,1}(f)| \leq \frac{2873}{2304}.$$

7. Conclusions

In this work, our main emphasis has been to highlight the significance and role of Fractional Calculus in Geometric Function Theory. For this purpose, we used the well-known q -fractional differential operator to define the q -analog of the cosine function. With the blend of these two concepts, we introduced the q -version of starlike functions, followed by defining the same function in the domain of the q -cosine function. This association was made successful using the concept of differential subordination. Working classically, we explored and investigated this class of functions by establishing the coefficient inequalities, including the coefficient bounds, Zalcman inequalities, and the upper bounds of the Hankel and the Toeplitz determinants. It is pertinent to mention that the obtained results, when compared with already-known results for simpler versions of these functions, were found to be much better and improved. Many other interesting dimensions are still to be explored, and we strongly believe that this work can be used as a reference for many more to come.

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