

Article

LPV H_∞ Control with an Augmented Nonlinear Observer for Sawyer Motors

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Abstract: This study presents LPV H_∞ control with an augmented nonlinear observer (ANOB) to improve both the position and yaw tracking errors for Sawyer motors. The proposed control method consists of the forces and torque modulation scheme, an ANOB, and a Lyapunov-based current controller with the LPV H_∞ state feedback controller to guarantee the stability of tracking error dynamics. The ANOB is designed to estimate all the state variables including the position, velocity, current, and disturbance using only position feedback. We propose a vertex expansion technique to solve the influence of the convex interpolation parameters in the LPV system on the tracking error performance. To be robust against disturbance, a state feedback controller with the LPV gain scheduling is determined by applying the H_∞ control in the linear-matrix-inequality (LMI) technique. The closed-loop stability is proved through the Lyapunov theory. The effectiveness of the proposed control method is evaluated through simulation results and compared with the conventional proportional-integral-derivative (PID) control method to verify both the improved tracking error performance and a suitable disturbance rejection.

Keywords: Sawyer motors; linear parameter varying (LPV) system; disturbance observer; augmented observer; linear matrix inequality (LMI); H_∞ control



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1. Introduction

In high-precision motion control, Sawyer motors have been widely used to achieve the strict requirements in position control. These motors play an extremely important role in automated assembly lines, precision machining tools, and semiconductor wafer stage systems. The Sawyer motors are dual-axis linear motors that have four force generators mounted symmetrically on the housing part [1]. Force generators X1 and X2 generate force in the X direction while force generators Y1 and Y2 generate force in the Y direction, as shown in Figure 1. Yaw of the motor is generated by asymmetries in the force generators and leads to loss of synchronization between the motor and platen teeth. The motor is not only a high-order dynamics but also has complex nonlinear properties. In addition, the motor has high position resolution and speed motion in the open-loop microstepping control. However, it can have step-out, a long settling time, and especially low disturbance compensation. These problems are closely related to the yaw motion control that degrades the position tracking performance. Thus, the motor requires a highly accurate position control method.

Many advanced control approaches have been proposed to improve the position tracking performance for the Sawyer motors [1–5]. Structures of various robust adaptive controllers have been developed to guarantee position tracking performance [1,2]. To achieve the desired position, the nonlinear cascade control algorithms used linear techniques, such as conventional PID control [3]. By modifying proportional-integral-derivative (PID) and lead-and-PID compensators, they are designed to obtain the desired performances [4]. A combination of robust adaptive control and iterative learning control make compensation

decisions to adapt to uncertain parameters and improve transient response in the position control [5]. However, the above-mentioned control algorithms are not optimal in the position tracking performance because the optimal control of nonlinear systems is very difficult and a great challenge. Although the optimal control of these motors was also studied to improve the positioning precision [6,7], the performance criteria for nonlinear dynamics are still uncertain and the influence of disturbance in the position control has not been considered; disturbance plays a critical role and directly affects control of the yaw angle of the motor. In addition, complex nonlinear models of the systems always contain nonlinear parameters that vary throughout the control process, making the nonlinear control system unstable. Therefore, linear parameter varying (LPV) systems whose nonlinear parameters are updated continuously online through measurements were developed to improve the control quality through greater stability and robustness [8–10].

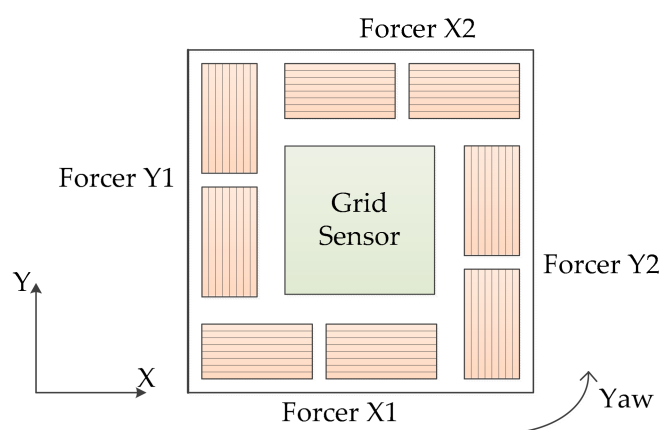


Figure 1. Bottom view of a Sawyer motor.

To address the influence of disturbance, many different disturbance observers (DOBs) were designed to estimate and compensate for disturbance [11–19]. DOB was generalized to obtain the high-order disturbances in the sense of the time series expansion [11]. The use of all the measured states and linear DOB techniques based on control algorithms to reject external disturbance was studied in [12–15]. To improve the transient response by compensating for unknown and uncertain disturbances, a state-space DOB is proposed in [16,17]. However, the limitation of the aforementioned DOBs needs accurate state information regarding the nonlinear dynamics to estimate disturbance. This limitation is extremely difficult in practice to measure all the state variables using sensors because there are several problems, such as cost and space limitations. Therefore, an augmented nonlinear observer (ANOB) design is considered to estimate all the state variables including disturbance. In addition, there are different strategies to compensate the friction using the friction model and comparing the output of the inverse model and the input of the original nonlinear system [18,19]; however, it is difficult for friction compensation because the friction is very diverse. In the context of the previous studies mentioned above, the augmented nonlinear observer design is essential for the Sawyer motors to improve the transient response in position control in the presence of disturbance.

The main contribution of this study was to design the LPV H_∞ control with an ANOB to reduce both the position and yaw tracking errors for the Sawyer motors in the presence of disturbance. The proposed control method consists of the forces and torque modulation scheme, an ANOB, and a Lyapunov-based current controller with the LPV H_∞ state feedback controller. The forces and torque modulation scheme is developed to guarantee the stability of the position and yaw tracking errors for the mechanical tracking error dynamics and to generate the desired phase currents. A Lyapunov-based current controller with an LPV H_∞ state feedback controller is proposed to guarantee the stability of phase current tracking errors for the electrical tracking error dynamics under the influence of back-electromotive forces (EMFs) and phase lag by inductances

during operation. All the state variables (position, velocity, current) including the load forces and load torque are estimated by using only the position feedback. Estimated state variables are used instead of the measured signals. This makes the control performance unaffected under the measurement noise in the currents. The position control problem of the Sawyer motor was analyzed in the optimal LPV problem. A vertex expansion technique is introduced to compute the convex interpolation parameters to solve the influence of vertices on the tracking performance in the LPV system. To be robust against disturbance, a state feedback controller with the LPV gain scheduling is determined by applying the H_∞ control in the linear-matrix-inequality (LMI) technique. The closed-loop stability is proved through the Lyapunov theory. The effectiveness of the proposed control method was demonstrated by comparing the simulation results with the performance of the conventional PID control method.

This article is organized into seven sections. Section 2 describes the mathematical model of the Sawyer motor. Section 3 defines the forces, torque modulation scheme, and the Lyapunov-based current controller. Section 4 defines the configuration of the LPV system. Section 5 proposes the ANOB and H_∞ feedback control design using LPV synthesis including the ANOB and LPV H_∞ state feedback Control. Section 6 presents the simulation results of the proposed control method and a performance comparison with the conventional PID control method. Finally, Section 7 presents the conclusions.

2. Mathematical Model of Sawyer Motor

The mathematical model of the Sawyer motor consists of mechanical and electrical dynamics that can be respectively represented [1,2] as follows:

$$\begin{aligned}\dot{\theta}_x &= \omega_x \\ \dot{\omega}_x &= \frac{-B_x \omega_x + F_{x1} + F_{x2} - d_x}{M} \\ \dot{\theta}_y &= \omega_y \\ \dot{\omega}_y &= \frac{-B_y \omega_y + F_{y1} + F_{y2} - d_y}{M}\end{aligned}\quad (1)$$

$$\begin{aligned}\dot{\theta}_{yaw} &= \omega_{yaw} \\ \dot{\omega}_{yaw} &= \frac{-B_{yaw} \omega_{yaw} + I_x (F_{x1} - F_{x2}) + I_y (F_{y1} - F_{y2}) - d_{yaw}}{J} \\ \dot{i}_{x1a} &= \frac{-R i_{x1a} + \kappa \sin(\gamma \theta_{x1}) (\omega_x + I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x1a}}{L} \\ \dot{i}_{x1b} &= \frac{-R i_{x1b} - \kappa \cos(\gamma \theta_{x1}) (\omega_x + I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x1b}}{L} \\ \dot{i}_{x2a} &= \frac{-R i_{x2a} + \kappa \sin(\gamma \theta_{x2}) (\omega_x - I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x2a}}{L} \\ \dot{i}_{x2b} &= \frac{-R i_{x2b} - \kappa \cos(\gamma \theta_{x2}) (\omega_x - I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x2b}}{L} \\ \dot{i}_{y1a} &= \frac{-R i_{y1a} + \kappa \sin(\gamma \theta_{y1}) (\omega_y + I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y1a}}{L} \\ \dot{i}_{y1b} &= \frac{-R i_{y1b} - \kappa \cos(\gamma \theta_{y1}) (\omega_y + I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y1b}}{L} \\ \dot{i}_{y2a} &= \frac{-R i_{y2a} + \kappa \sin(\gamma \theta_{y2}) (\omega_y - I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y2a}}{L} \\ \dot{i}_{y2b} &= \frac{-R i_{y2b} - \kappa \cos(\gamma \theta_{y2}) (\omega_y - I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y2b}}{L}\end{aligned}\quad (2)$$

where $\gamma = \frac{2\pi}{p}$ and

$$\begin{aligned} F_{x1} &= -\kappa \sin(\gamma\theta_{x1})i_{x1a} + \kappa \cos(\gamma\theta_{x1})i_{x1b}, F_{x2} = -\kappa \sin(\gamma\theta_{x2})i_{x2a} + \kappa \cos(\gamma\theta_{x2})i_{x2b} \\ F_{y1} &= -\kappa \sin(\gamma\theta_{y1})i_{y1a} + \kappa \cos(\gamma\theta_{y1})i_{y1b}, F_{y2} = -\kappa \sin(\gamma\theta_{y2})i_{y2a} + \kappa \cos(\gamma\theta_{y2})i_{y2b} \\ \theta_{x1} &= \theta_x + l_x \sin(\theta_{yaw}), \theta_{x2} = \theta_x - l_x \sin(\theta_{yaw}) \\ \theta_{y1} &= \theta_y + l_y \sin(\theta_{yaw}), \theta_{y2} = \theta_y - l_y \sin(\theta_{yaw}) \end{aligned} \quad (3)$$

where θ_x , θ_y , and θ_{yaw} are the positions of the X-axis, Y-axis, and yaw rotation of the motor, respectively; ω_x , ω_y , and ω_{yaw} are the velocities of the X-axis, Y-axis, and yaw rate of the motor, respectively; l_x and l_y are the distances from the center of the motor to the forcer; F_{x1} and F_{x2} are forces made by the two X-axis forcers, whereas F_{y1} and F_{y2} are forces made by the two Y-axis forcers; κ is the force constant; M and J denote the motor mass and inertia, respectively; B_x , B_y , and B_{yaw} are coefficients of friction; p is the tooth pitch of the platen; i_{x1a} , i_{x1b} , i_{x2a} , i_{x2b} , i_{y1a} , i_{y1b} , i_{y2a} , and i_{y2b} are phase currents; v_{x1a} , v_{x1b} , v_{x2a} , v_{x2b} , v_{y1a} , v_{y1b} , v_{y2a} , and v_{y2b} are input voltages in phases A and B; and d_x , d_y , and d_{yaw} are unknown load forces and torque, respectively.

3. Forces, Torque Modulation Scheme, and Lyapunov-Based Current Controller

3.1. Tracking Error Dynamics

Let us define the tracking-error state vectors as follows:

$$\begin{aligned} e &= \begin{bmatrix} e_{mech}^T & e_{elec}^T \end{bmatrix}^T \\ e_{mech} &= \begin{bmatrix} e_{\theta_x} & e_{\omega_x} & e_{\theta_y} & e_{\omega_y} & e_{\theta_{yaw}} & e_{\omega_{yaw}} \end{bmatrix}^T \\ e_{elec} &= \begin{bmatrix} e_{x1a} & e_{x1b} & e_{x2a} & e_{x2b} & e_{y1a} & e_{y1b} & e_{y2a} & e_{y2b} \end{bmatrix}^T \end{aligned} \quad (4)$$

where:

$$\begin{aligned} e_{\theta_x} &= \theta_x^d - \theta_x, \quad e_{\omega_x} = \omega_x^* - \omega_x \\ e_{\theta_y} &= \theta_y^d - \theta_y, \quad e_{\omega_y} = \omega_y^* - \omega_y \\ e_{\theta_{yaw}} &= \theta_{yaw}^d - \theta_{yaw}, \quad e_{\omega_{yaw}} = \omega_{yaw}^* - \omega_{yaw} \\ e_{x1a} &= i_{x1a}^d - i_{x1a}, \quad e_{x1b} = i_{x1b}^d - i_{x1b} \\ e_{x2a} &= i_{x2a}^d - i_{x2a}, \quad e_{x2b} = i_{x2b}^d - i_{x2b} \\ e_{y1a} &= i_{y1a}^d - i_{y1a}, \quad e_{y1b} = i_{y1b}^d - i_{y1b} \\ e_{y2a} &= i_{y2a}^d - i_{y2a}, \quad e_{y2b} = i_{y2b}^d - i_{y2b} \end{aligned} \quad (5)$$

where θ_x^d , θ_y^d , and θ_{yaw}^d are the desired positions of the X-axis, Y-axis, and yaw rotation, respectively; ω_x^* , ω_y^* , and ω_{yaw}^* are the virtual velocities of the X-axis, Y-axis, and virtual yaw rate, respectively; e_{θ_x} , e_{θ_y} , and $e_{\theta_{yaw}}$ are the position tracking errors of the X-axis, Y-axis, and yaw rotation, respectively; e_{ω_x} , e_{ω_y} , and $e_{\omega_{yaw}}$ are the velocity tracking errors of the X-axis, Y-axis, and yaw rate, respectively; i_{x1a}^d , i_{x1b}^d , i_{x2a}^d , i_{x2b}^d , i_{y1a}^d , i_{y1b}^d , i_{y2a}^d , and i_{y2b}^d are the desired phase currents; and e_{x1a} , e_{x1b} , e_{x2a} , e_{x2b} , e_{y1a} , e_{y1b} , e_{y2a} , and e_{y2b} are the current tracking errors in phases A and B.

Differentiating (5) with respect to time, the tracking error dynamics are represented as follows:

$$\begin{aligned}
 \dot{e}_{\theta_x} &= \omega_x^d - \omega_x \\
 \dot{e}_{\omega_x} &= \frac{1}{M} \begin{pmatrix} M\dot{\omega}_x^* + B_x\omega_x + \kappa \sin(\gamma\theta_{x1})i_{x1a} - \kappa \cos(\gamma\theta_{x1})i_{x1b} \\ + \kappa \sin(\gamma\theta_{x2})i_{x2a} - \kappa \cos(\gamma\theta_{x2})i_{x2b} + d_x \end{pmatrix} \\
 \dot{e}_{\theta_y} &= \omega_y^d - \omega_y \\
 \dot{e}_{\omega_y} &= \frac{1}{M} \begin{pmatrix} M\dot{\omega}_y^* + B_y\omega_y + \kappa \sin(\gamma\theta_{y1})i_{y1a} - \kappa \cos(\gamma\theta_{y1})i_{y1b} \\ + \kappa \sin(\gamma\theta_{y2})i_{y2a} - \kappa \cos(\gamma\theta_{y2})i_{y2b} + d_y \end{pmatrix} \\
 \dot{e}_{\theta_{yaw}} &= \omega_{yaw}^d - \omega_{yaw} \\
 \dot{e}_{\omega_{yaw}} &= \frac{1}{J} \begin{pmatrix} J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw} + \kappa l_x \sin(\gamma\theta_{x1})i_{x1a} \\ - \kappa l_x \cos(\gamma\theta_{x1})i_{x1b} - \kappa l_x \sin(\gamma\theta_{x2})i_{x2a} \\ + \kappa l_x \cos(\gamma\theta_{x2})i_{x2b} + \kappa l_y \sin(\gamma\theta_{y1})i_{y1a} \\ - \kappa l_y \cos(\gamma\theta_{y1})i_{y1b} - \kappa l_y \sin(\gamma\theta_{y2})i_{y2a} \\ + \kappa l_y \cos(\gamma\theta_{y2})i_{y2b} + d_{yaw} \end{pmatrix}
 \end{aligned} \tag{6}$$

where $\omega_x^d = \dot{\theta}_x^d$, $\omega_y^d = \dot{\theta}_y^d$, and $\omega_{yaw}^d = \dot{\theta}_{yaw}^d$.

$$\begin{aligned}
 \dot{e}_{x1a} &= \frac{1}{L} \left(L\dot{i}_{x1a}^d + Ri_{x1a} - \kappa \sin(\gamma\theta_{x1})(\omega_x + l_x \cos(\theta_{yaw})\omega_{yaw}) - v_{x1a} \right) \\
 \dot{e}_{x1b} &= \frac{1}{L} \left(L\dot{i}_{x1b}^d + Ri_{x1b} + \kappa \cos(\gamma\theta_{x1})(\omega_x + l_x \cos(\theta_{yaw})\omega_{yaw}) - v_{x1b} \right) \\
 \dot{e}_{x2a} &= \frac{1}{L} \left(L\dot{i}_{x2a}^d + Ri_{x2a} - \kappa \sin(\gamma\theta_{x2})(\omega_x - l_x \cos(\theta_{yaw})\omega_{yaw}) - v_{x2a} \right) \\
 \dot{e}_{x2b} &= \frac{1}{L} \left(L\dot{i}_{x2b}^d + Ri_{x2b} + \kappa \cos(\gamma\theta_{x2})(\omega_x - l_x \cos(\theta_{yaw})\omega_{yaw}) - v_{x2b} \right) \\
 \dot{e}_{y1a} &= \frac{1}{L} \left(L\dot{i}_{y1a}^d + Ri_{y1a} - \kappa \sin(\gamma\theta_{y1})(\omega_y + l_y \cos(\theta_{yaw})\omega_{yaw}) - v_{y1a} \right) \\
 \dot{e}_{y1b} &= \frac{1}{L} \left(L\dot{i}_{y1b}^d + Ri_{y1b} + \kappa \cos(\gamma\theta_{y1})(\omega_y + l_y \cos(\theta_{yaw})\omega_{yaw}) - v_{y1b} \right) \\
 \dot{e}_{y2a} &= \frac{1}{L} \left(L\dot{i}_{y2a}^d + Ri_{y2a} - \kappa \sin(\gamma\theta_{y2})(\omega_y - l_y \cos(\theta_{yaw})\omega_{yaw}) - v_{y2a} \right) \\
 \dot{e}_{y2b} &= \frac{1}{L} \left(L\dot{i}_{y2b}^d + Ri_{y2b} + \kappa \cos(\gamma\theta_{y2})(\omega_y - l_y \cos(\theta_{yaw})\omega_{yaw}) - v_{y2b} \right)
 \end{aligned} \tag{7}$$

3.2. Forces and Torque Modulation Scheme

To guarantee the stability of the position and yaw tracking errors for the mechanical tracking error dynamics, we propose to express the forces and torque modulation in the LPV system as follows:

$$\begin{aligned}
 \omega_x^* &= \omega_x^d + k_{\theta_x} e_{\theta_x}, \omega_y^* = \omega_y^d + k_{\theta_y} e_{\theta_y}, \omega_{yaw}^* = \omega_{yaw}^d + k_{\theta_{yaw}} e_{\theta_{yaw}} \\
 F_{x1}^d &= F_{x1}^d + F_{x2}^d = M\dot{\omega}_x^* + B_x\omega_x^* + e_{\theta_x} + k_{\omega_x} e_{\omega_x} + d_x \\
 F_y^d &= F_{y1}^d + F_{y2}^d = M\dot{\omega}_y^* + B_y\omega_y^* + e_{\theta_y} + k_{\omega_y} e_{\omega_y} + d_y \\
 \tau_{yaw}^d &= l_x (F_{x1}^d - F_{x2}^d) + l_y (F_{y1}^d - F_{y2}^d) \\
 &= J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}} e_{\omega_{yaw}} + d_{yaw}
 \end{aligned} \tag{8}$$

where $k_{\theta_x}, k_{\theta_y}, k_{\theta_{yaw}}, k_{\omega_x}, k_{\omega_y}$, and $k_{\omega_{yaw}}$ are positive constants.

We define the commutation scheme in the LPV system to generate the desired phase currents as follows:

$$\begin{aligned}
 i_{x1a}^d &= -\left(\frac{F_x^d}{2\kappa} + \frac{\tau_{yaw}^d}{4\kappa l_x}\right) \sin(\gamma\theta_{x1}) \\
 &= -\left(\frac{\frac{M\dot{\omega}_x^* + B_x\omega_x^* + e_{\theta_x} + k_{\omega_x}e_{\omega_x} + d_x}{2\kappa}}{\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_x}}\right) \sin(\gamma\theta_{x1}) \\
 i_{x1b}^d &= \left(\frac{F_x^d}{2\kappa} + \frac{\tau_{yaw}^d}{4\kappa l_x}\right) \cos(\gamma\theta_{x1}) \\
 &= \left(\frac{\frac{M\dot{\omega}_x^* + B_x\omega_x^* + e_{\theta_x} + k_{\omega_x}e_{\omega_x} + d_x}{2\kappa}}{\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_x}}\right) \cos(\gamma\theta_{x1})
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 i_{x2a}^d &= -\left(\frac{F_x^d}{2\kappa} - \frac{\tau_{yaw}^d}{4\kappa l_x}\right) \sin(\gamma\theta_{x2}) \\
 &= -\left(\frac{\frac{M\dot{\omega}_x^* + B_x\omega_x^* + e_{\theta_x} + k_{\omega_x}e_{\omega_x} + d_x}{2\kappa}}{-\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_x}}\right) \sin(\gamma\theta_{x2}) \\
 i_{x2b}^d &= \left(\frac{F_x^d}{2\kappa} - \frac{\tau_{yaw}^d}{4\kappa l_x}\right) \cos(\gamma\theta_{x2}) \\
 &= \left(\frac{\frac{M\dot{\omega}_x^* + B_x\omega_x^* + e_{\theta_x} + k_{\omega_x}e_{\omega_x} + d_x}{2\kappa}}{-\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_x}}\right) \cos(\gamma\theta_{x2})
 \end{aligned}$$

$$\begin{aligned}
 i_{y1a}^d &= -\left(\frac{F_y^d}{2\kappa} + \frac{\tau_{yaw}^d}{4\kappa l_y}\right) \sin(\gamma\theta_{y1}) \\
 &= -\left(\frac{\frac{M\dot{\omega}_y^* + B_y\omega_y^* + e_{\theta_y} + k_{\omega_y}e_{\omega_y} + d_y}{2\kappa}}{\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_y}}\right) \sin(\gamma\theta_{y1}) \\
 i_{y1b}^d &= \left(\frac{F_y^d}{2\kappa} + \frac{\tau_{yaw}^d}{4\kappa l_y}\right) \cos(\gamma\theta_{y1}) \\
 &= \left(\frac{\frac{M\dot{\omega}_y^* + B_y\omega_y^* + e_{\theta_y} + k_{\omega_y}e_{\omega_y} + d_y}{2\kappa}}{\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_y}}\right) \cos(\gamma\theta_{y1})
 \end{aligned} \tag{10}$$

$$\begin{aligned}
 i_{y2a}^d &= -\left(\frac{F_y^d}{2\kappa} - \frac{\tau_{yaw}^d}{4\kappa l_y}\right) \sin(\gamma\theta_{y2}) \\
 &= -\left(\frac{\frac{M\dot{\omega}_y^* + B_y\omega_y^* + e_{\theta_y} + k_{\omega_y}e_{\omega_y} + d_y}{2\kappa}}{-\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_y}}\right) \sin(\gamma\theta_{y2}) \\
 i_{y2b}^d &= \left(\frac{F_y^d}{2\kappa} - \frac{\tau_{yaw}^d}{4\kappa l_y}\right) \cos(\gamma\theta_{y2}) \\
 &= \left(\frac{\frac{M\dot{\omega}_y^* + B_y\omega_y^* + e_{\theta_y} + k_{\omega_y}e_{\omega_y} + d_y}{2\kappa}}{-\frac{J\dot{\omega}_{yaw}^* + B_{yaw}\omega_{yaw}^* + e_{\theta_{yaw}} + k_{\omega_{yaw}}e_{\omega_{yaw}} + d_{yaw}}{4\kappa l_y}}\right) \cos(\gamma\theta_{y2})
 \end{aligned} \tag{11}$$

3.3. Lyapunov-Based Current Controller

To guarantee the stability of phase current tracking errors for the electrical tracking error dynamics under the influence of back-EMFs and phase lag by inductances during operation, we propose the Lyapunov-based current controller with LPV H_∞ state feedback control in the LPV system in terms of Theorem 1.

Theorem 1. According to the tracking error dynamics expressed by (6) and (7), the voltage inputs of the Sawyer motor are defined as follows:

$$\begin{aligned}
 v_{x1a} &= \dot{L}_{x1a}^d + R_{x1a}^d - \kappa \omega_x^* \sin(\gamma \theta_{x1}) - \kappa l_x \sin(\gamma \theta_{x1}) e_{\omega_{yaw}} \\
 &\quad - \kappa l_x \cos(\theta_{yaw}) \sin(\gamma \theta_{x1}) \omega_{yaw} + u_{x1a} \\
 v_{x1b} &= \dot{L}_{x1b}^d + R_{x1b}^d + \kappa \omega_x^* \cos(\gamma \theta_{x1}) + \kappa l_x \cos(\gamma \theta_{x1}) e_{\omega_{yaw}} \\
 &\quad + \kappa l_x \cos(\theta_{yaw}) \cos(\gamma \theta_{x1}) \omega_{yaw} + u_{x1b} \\
 v_{x2a} &= \dot{L}_{x2a}^d + R_{x2a}^d - \kappa \omega_x^* \sin(\gamma \theta_{x2}) + \kappa l_x \sin(\gamma \theta_{x2}) e_{\omega_{yaw}} \\
 &\quad + \kappa l_x \cos(\theta_{yaw}) \sin(\gamma \theta_{x2}) \omega_{yaw} + u_{x2a} \\
 v_{x2b} &= \dot{L}_{x2b}^d + R_{x2b}^d + \kappa \omega_x^* \cos(\gamma \theta_{x2}) - \kappa l_x \cos(\gamma \theta_{x2}) e_{\omega_{yaw}} \\
 &\quad - \kappa l_x \cos(\theta_{yaw}) \cos(\gamma \theta_{x2}) \omega_{yaw} + u_{x2b} \\
 v_{y1a} &= \dot{L}_{y1a}^d + R_{y1a}^d - \kappa \omega_y^* \sin(\gamma \theta_{y1}) - \kappa l_y \sin(\gamma \theta_{y1}) e_{\omega_{yaw}} \\
 &\quad - \kappa l_y \cos(\theta_{yaw}) \sin(\gamma \theta_{y1}) \omega_{yaw} + u_{y1a} \\
 v_{y1b} &= \dot{L}_{y1b}^d + R_{y1b}^d + \kappa \omega_y^* \cos(\gamma \theta_{y1}) + \kappa l_y \cos(\gamma \theta_{y1}) e_{\omega_{yaw}} \\
 &\quad + \kappa l_y \cos(\theta_{yaw}) \cos(\gamma \theta_{y1}) \omega_{yaw} + u_{y1b} \\
 v_{y2a} &= \dot{L}_{y2a}^d + R_{y2a}^d - \kappa \omega_y^* \sin(\gamma \theta_{y2}) + \kappa l_y \sin(\gamma \theta_{y2}) e_{\omega_{yaw}} \\
 &\quad + \kappa l_y \cos(\theta_{yaw}) \sin(\gamma \theta_{y2}) \omega_{yaw} + u_{y2a} \\
 v_{y2b} &= \dot{L}_{y2b}^d + R_{y2b}^d + \kappa \omega_y^* \cos(\gamma \theta_{y2}) - \kappa l_y \cos(\gamma \theta_{y2}) e_{\omega_{yaw}} \\
 &\quad - \kappa l_y \cos(\theta_{yaw}) \cos(\gamma \theta_{y2}) \omega_{yaw} + u_{y2b}
 \end{aligned} \tag{12}$$

where the H_∞ state feedback control law with LPV gain scheduling K for the electrical tracking error dynamics, whose design is presented in Section 5, is expressed as follows:

$$\begin{aligned}
 u &= Ke, \quad K \in \mathbb{R}^{8 \times 14}, \quad K = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_8 \end{bmatrix} = \begin{bmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \\ \vdots & \vdots \\ K_{8,1} & K_{8,2} \end{bmatrix} \\
 u &= [u_{x1a} \quad u_{x1b} \quad u_{x2a} \quad u_{x2b} \quad u_{y1a} \quad u_{y1b} \quad u_{y2a} \quad u_{y2b}]^T \\
 K_{1,1} &= [k_{1,1} \quad k_{1,2} \quad \dots \quad k_{1,6}], \quad K_{1,2} = [k_{1,7} \quad k_{1,8} \quad \dots \quad k_{1,14}] \\
 K_{2,1} &= [k_{2,1} \quad k_{2,2} \quad \dots \quad k_{2,6}], \quad K_{2,2} = [k_{2,7} \quad k_{2,8} \quad \dots \quad k_{2,14}] \\
 &\vdots \\
 K_{8,1} &= [k_{8,1} \quad k_{8,2} \quad \dots \quad k_{8,6}], \quad K_{8,2} = [k_{8,7} \quad k_{8,8} \quad \dots \quad k_{8,14}]
 \end{aligned} \tag{13}$$

Proof of Theorem 1. Let us consider the first Lyapunov function candidate:

$$V_1 = \frac{1}{2} e_{\theta_x}^2 + \frac{1}{2} e_{\theta_y}^2 + \frac{1}{2} e_{\theta_{yaw}}^2 \tag{14}$$

The derivative of V_1 with respect to time is expressed as follows:

$$\begin{aligned}\dot{V}_1 &= e_{\theta_x} \dot{e}_{\theta_x} + e_{\theta_y} \dot{e}_{\theta_y} + e_{\theta_{yaw}} \dot{e}_{\theta_{yaw}} \\ &= -k_{\theta_x} e_{\theta_x}^2 - k_{\theta_y} e_{\theta_y}^2 - k_{\theta_{yaw}} e_{\theta_{yaw}}^2 + e_{\theta_x} e_{\omega_x} + e_{\theta_y} e_{\omega_y} + e_{\theta_{yaw}} e_{\omega_{yaw}}\end{aligned}$$

Let us consider now the second Lyapunov function candidate:

$$V_2 = V_1 + \frac{1}{2} M e_{\omega_x}^2 + \frac{1}{2} M e_{\omega_y}^2 + \frac{1}{2} J e_{\omega_{yaw}}^2 \quad (15)$$

Using (8), the derivative of V_2 with respect to time is obtained as follows:

$$\begin{aligned}\dot{V}_2 &= -k_{\theta_x} e_{\theta_x}^2 - k_{\theta_y} e_{\theta_y}^2 - k_{\theta_{yaw}} e_{\theta_{yaw}}^2 \\ &\quad - (k_{\omega_x} + B_x) e_{\omega_x}^2 - (k_{\omega_y} + B_y) e_{\omega_y}^2 - (k_{\omega_{yaw}} + B_{yaw}) e_{\omega_{yaw}}^2 \\ &\quad + e_{\omega_x} \begin{pmatrix} -\kappa \sin(\gamma_{\theta_{x1}}) e_{x1a} + \kappa \cos(\gamma_{\theta_{x1}}) e_{x1b} \\ -\kappa \sin(\gamma_{\theta_{x2}}) e_{x2a} + \kappa \cos(\gamma_{\theta_{x2}}) e_{x2b} \end{pmatrix} \\ &\quad + e_{\omega_y} \begin{pmatrix} -\kappa \sin(\gamma_{\theta_{y1}}) e_{y1a} + \kappa \cos(\gamma_{\theta_{y1}}) e_{y1b} \\ -\kappa \sin(\gamma_{\theta_{y2}}) e_{y2a} + \kappa \cos(\gamma_{\theta_{y2}}) e_{y2b} \end{pmatrix} \\ &\quad + e_{\omega_{yaw}} \begin{pmatrix} -\kappa l_x \sin(\gamma_{\theta_{x1}}) e_{x1a} + \kappa l_x \cos(\gamma_{\theta_{x1}}) e_{x1b} \\ + \kappa l_x \sin(\gamma_{\theta_{x2}}) e_{x2a} - \kappa l_x \cos(\gamma_{\theta_{x2}}) e_{x2b} \\ - \kappa l_y \sin(\gamma_{\theta_{y1}}) e_{y1a} + \kappa l_y \cos(\gamma_{\theta_{y1}}) e_{y1b} \\ + \kappa l_y \sin(\gamma_{\theta_{y2}}) e_{y2a} - \kappa l_y \cos(\gamma_{\theta_{y2}}) e_{y2b} \end{pmatrix}\end{aligned}$$

Let us finally consider the third Lyapunov function candidate:

$$\begin{aligned}V_3 &= V_1 + V_2 + \frac{1}{2} L e_{x1a}^2 + \frac{1}{2} L e_{x1b}^2 + \frac{1}{2} L e_{x2a}^2 + \frac{1}{2} L e_{x2b}^2 \\ &\quad + \frac{1}{2} L e_{y1a}^2 + \frac{1}{2} L e_{y1b}^2 + \frac{1}{2} L e_{y2a}^2 + \frac{1}{2} L e_{y2b}^2\end{aligned} \quad (16)$$

Using v_{x1a} , v_{x1b} , v_{x2a} , v_{x2b} , v_{y1a} , v_{y1b} , v_{y2a} , and v_{y2b} in (12), and the LPV H_∞ state feedback control law with LPV gain scheduling K given by (13), the derivative of V_3 with respect to time is obtained as follows:

$$\begin{aligned}\dot{V}_3 &= -k_{\theta_x} e_{\theta_x}^2 - k_{\theta_y} e_{\theta_y}^2 - k_{\theta_{yaw}} e_{\theta_{yaw}}^2 - (k_{\omega_x} + B_x) e_{\omega_x}^2 - (k_{\omega_y} + B_y) e_{\omega_y}^2 \\ &\quad - (k_{\omega_{yaw}} + B_{yaw}) e_{\omega_{yaw}}^2 - (R + k_{1,7}) e_{x1a}^2 - (R + k_{2,8}) e_{x1b}^2 - (R + k_{3,9}) e_{x2a}^2 \\ &\quad - (R + k_{4,10}) e_{x2b}^2 - (R + k_{5,11}) e_{y1a}^2 - (R + k_{6,12}) e_{y1b}^2 - (R + k_{7,13}) e_{y2a}^2 \\ &\quad - (R + k_{8,14}) e_{y2b}^2 - e_{x1a} K_{1,1} e_{mech} - e_{x1b} K_{2,1} e_{mech} - e_{x2a} K_{3,1} e_{mech} - e_{x2b} K_{4,1} e_{mech} \\ &\quad - e_{y1a} K_{5,1} e_{mech} - e_{y1b} K_{6,1} e_{mech} - e_{y2a} K_{7,1} e_{mech} - e_{y2b} K_{8,1} e_{mech} - (k_{1,8} + k_{2,7}) e_{x1a} e_{x1b} \\ &\quad - (k_{1,9} + k_{3,7}) e_{x1a} e_{x2a} - (k_{1,10} + k_{4,7}) e_{x1a} e_{x2b} - (k_{1,11} + k_{5,7}) e_{x1a} e_{y1a} \\ &\quad - (k_{1,12} + k_{6,7}) e_{x1a} e_{y1b} - (k_{1,13} + k_{7,7}) e_{x1a} e_{y2a} - (k_{1,14} + k_{8,7}) e_{x1a} e_{y2b} \\ &\quad - (k_{2,9} + k_{3,8}) e_{x1b} e_{x2a} - (k_{2,10} + k_{4,8}) e_{x1b} e_{x2b} - (k_{2,11} + k_{5,8}) e_{x1b} e_{y1a} \\ &\quad - (k_{2,12} + k_{6,8}) e_{x1b} e_{y1b} - (k_{2,13} + k_{7,8}) e_{x1b} e_{y2a} - (k_{2,14} + k_{8,8}) e_{x1b} e_{y2b} \\ &\quad - (k_{3,10} + k_{4,9}) e_{x2a} e_{x2b} - (k_{3,11} + k_{5,9}) e_{x2a} e_{y1a} - (k_{3,12} + k_{6,9}) e_{x2a} e_{y1b} \\ &\quad - (k_{3,13} + k_{7,9}) e_{x2a} e_{y2a} - (k_{3,14} + k_{8,9}) e_{x2a} e_{y2b} - (k_{4,11} + k_{5,10}) e_{x2b} e_{y1a} \\ &\quad - (k_{4,12} + k_{6,10}) e_{x2b} e_{y1b} - (k_{4,13} + k_{7,10}) e_{x2b} e_{y2a} - (k_{4,14} + k_{8,10}) e_{x2b} e_{y2b} \\ &\quad - (k_{5,12} + k_{6,11}) e_{y1a} e_{y1b} - (k_{5,13} + k_{7,11}) e_{y1a} e_{y2a} - (k_{5,14} + k_{8,11}) e_{y1a} e_{y2b} \\ &\quad - (k_{6,13} + k_{7,12}) e_{y1b} e_{y2a} - (k_{6,14} + k_{8,12}) e_{y1b} e_{y2b} - (k_{7,14} + k_{8,13}) e_{y2a} e_{y2b}\end{aligned}$$

Let us introduce the following definition:

$$\dot{V}_3 = -e^T M e \quad (17)$$

$$\text{where } M = \begin{bmatrix} Q & S \\ S^T & F \end{bmatrix} \text{ and } F = \begin{bmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{bmatrix}$$

$$F_{1,1} = \begin{bmatrix} R + k_{1,7} & \frac{k_{1,8}+k_{2,7}}{2} & \frac{k_{1,9}+k_{3,7}}{2} & \frac{k_{1,10}+k_{4,7}}{2} \\ \frac{k_{1,8}+k_{2,7}}{2} & R + k_{2,8} & \frac{k_{2,9}+k_{3,8}}{2} & \frac{k_{2,10}+k_{4,8}}{2} \\ \frac{k_{1,9}+k_{3,7}}{2} & \frac{k_{2,9}+k_{3,8}}{2} & R + k_{3,9} & \frac{k_{3,10}+k_{4,9}}{2} \\ \frac{k_{1,10}+k_{4,7}}{2} & \frac{k_{2,10}+k_{4,8}}{2} & \frac{k_{3,10}+k_{4,9}}{2} & R + k_{4,10} \end{bmatrix}$$

$$F_{1,2} = \frac{1}{2} \begin{bmatrix} k_{1,11} + k_{5,7} & k_{1,12} + k_{6,7} & k_{1,13} + k_{7,7} & k_{1,14} + k_{8,7} \\ k_{2,11} + k_{5,8} & k_{2,12} + k_{6,8} & k_{2,13} + k_{7,8} & k_{2,14} + k_{8,8} \\ k_{3,11} + k_{5,9} & k_{3,12} + k_{6,9} & k_{3,13} + k_{7,9} & k_{3,14} + k_{8,9} \\ k_{4,11} + k_{5,10} & k_{4,12} + k_{6,10} & k_{4,13} + k_{7,10} & k_{4,14} + k_{8,10} \end{bmatrix}$$

$$F_{2,1} = \frac{1}{2} \begin{bmatrix} k_{1,11} + k_{5,7} & k_{2,11} + k_{5,8} & k_{3,11} + k_{5,9} & k_{4,11} + k_{5,10} \\ k_{1,12} + k_{6,7} & k_{2,12} + k_{6,8} & k_{3,12} + k_{6,9} & k_{4,12} + k_{6,10} \\ k_{1,13} + k_{7,7} & k_{2,13} + k_{7,8} & k_{3,13} + k_{7,9} & k_{4,13} + k_{7,10} \\ k_{1,14} + k_{8,7} & k_{2,14} + k_{8,8} & k_{3,14} + k_{8,9} & k_{4,14} + k_{8,10} \end{bmatrix}$$

$$F_{2,2} = \begin{bmatrix} R + k_{5,11} & \frac{k_{5,12}+k_{6,11}}{2} & \frac{k_{5,13}+k_{7,11}}{2} & \frac{k_{5,14}+k_{8,11}}{2} \\ \frac{k_{5,12}+k_{6,11}}{2} & R + k_{6,12} & \frac{k_{6,13}+k_{7,12}}{2} & \frac{k_{6,14}+k_{8,12}}{2} \\ \frac{k_{5,13}+k_{7,11}}{2} & \frac{k_{6,13}+k_{7,12}}{2} & R + k_{7,13} & \frac{k_{7,14}+k_{8,13}}{2} \\ \frac{k_{5,14}+k_{8,11}}{2} & \frac{k_{6,14}+k_{8,12}}{2} & \frac{k_{7,14}+k_{8,13}}{2} & R + k_{8,14} \end{bmatrix}$$

$$S = \frac{1}{2} \begin{bmatrix} K_{1,1}^T & K_{2,1}^T & K_{3,1}^T & K_{4,1}^T & K_{5,1}^T & K_{6,1}^T & K_{7,1}^T & K_{8,1}^T \end{bmatrix}$$

$$Q = \text{diag} \left[k_{\theta_x}, k_{\omega_x} + B_x, k_{\theta_y}, k_{\omega_y} + B_y, k_{\theta_{yaw}}, k_{\omega_{yaw}} + B_{yaw} \right]$$

According to the Schur complement [20], $Q > 0$ is equivalent to the following expression:

$$F - S^T Q^{-1} S > 0 \quad (18)$$

Given that $k_{\theta_x}, k_{\theta_y}, k_{\theta_{yaw}}, k_{\omega_x}, k_{\omega_y}, k_{\omega_{yaw}}$, and K always exists to satisfy the condition in (18), $\dot{V}_3$ is definitely negative. Therefore, $e(t)$ converges to zero and the closed-loop stability is guaranteed.

Substituting (9), (10), (11), and (12) into (6) and (7), the tracking error dynamics can be re-written as follows:

$$\begin{aligned}
 \dot{e}_{\theta_x} &= e_{\omega_x} - k_{\theta_x} e_{\theta_x} \\
 \dot{e}_{\omega_x} &= \frac{1}{M} \begin{pmatrix} -e_{\theta_x} - (k_{\omega_x} + B_x) e_{\omega_x} - \kappa \sin(\gamma\theta_{x1}) e_{x1a} + \kappa \cos(\gamma\theta_{x1}) e_{x1b} \\ -\kappa \sin(\gamma\theta_{x2}) e_{x2a} + \kappa \cos(\gamma\theta_{x2}) e_{x2b} \end{pmatrix} \\
 \dot{e}_{\theta_y} &= e_{\omega_y} - k_{\theta_y} e_{\theta_y} \\
 \dot{e}_{\omega_y} &= \frac{1}{M} \begin{pmatrix} -e_{\theta_y} - (k_{\omega_y} + B_y) e_{\omega_y} - \kappa \sin(\gamma\theta_{y1}) e_{y1a} + \kappa \cos(\gamma\theta_{y1}) e_{y1b} \\ -\kappa \sin(\gamma\theta_{y2}) e_{y2a} + \kappa \cos(\gamma\theta_{y2}) e_{y2b} \end{pmatrix} \\
 \dot{e}_{\theta_{yaw}} &= e_{\omega_{yaw}} - k_{\theta_{yaw}} e_{\theta_{yaw}} \\
 \dot{e}_{\omega_{yaw}} &= \frac{1}{J} \begin{pmatrix} -e_{\theta_{yaw}} - (k_{\omega_{yaw}} + B_{yaw}) e_{\omega_{yaw}} - \kappa \sin(\gamma\theta_{x1}) e_{x1a} + \kappa \cos(\gamma\theta_{x1}) e_{x1b} \\ +\kappa \sin(\gamma\theta_{x2}) e_{x2a} - \kappa \cos(\gamma\theta_{x2}) e_{x2b} - \kappa \sin(\gamma\theta_{y1}) e_{y1a} \\ +\kappa \cos(\gamma\theta_{y1}) e_{y1b} + \kappa \sin(\gamma\theta_{y2}) e_{y2a} - \kappa \cos(\gamma\theta_{y2}) e_{y2b} \end{pmatrix} \quad (19) \\
 \dot{e}_{x1a} &= \frac{1}{L} \left(\kappa \sin(\gamma\theta_{x1}) e_{\omega_x} - R e_{x1a} + \kappa l_x \sin(\gamma\theta_{x1}) e_{\omega_{yaw}} - u_{x1a} \right) \\
 \dot{e}_{x1b} &= \frac{1}{L} \left(-\kappa \cos(\gamma\theta_{x1}) e_{\omega_x} - R e_{x1b} - \kappa l_x \cos(\gamma\theta_{x1}) e_{\omega_{yaw}} - u_{x1b} \right) \\
 \dot{e}_{x2a} &= \frac{1}{L} \left(\kappa \sin(\gamma\theta_{x2}) e_{\omega_x} - R e_{x2a} - \kappa l_x \sin(\gamma\theta_{x2}) e_{\omega_{yaw}} - u_{x2a} \right) \\
 \dot{e}_{x2b} &= \frac{1}{L} \left(-\kappa \cos(\gamma\theta_{x2}) e_{\omega_x} - R e_{x2b} + \kappa l_x \cos(\gamma\theta_{x2}) e_{\omega_{yaw}} - u_{x2b} \right) \\
 \dot{e}_{y1a} &= \frac{1}{L} \left(\kappa \sin(\gamma\theta_{y1}) e_{\omega_y} - R e_{y1a} + \kappa l_y \sin(\gamma\theta_{y1}) e_{\omega_{yaw}} - u_{y1a} \right) \\
 \dot{e}_{y1b} &= \frac{1}{L} \left(-\kappa \cos(\gamma\theta_{y1}) e_{\omega_y} - R e_{y1b} - \kappa l_y \cos(\gamma\theta_{y1}) e_{\omega_{yaw}} - u_{y1b} \right) \\
 \dot{e}_{y2a} &= \frac{1}{L} \left(\kappa \sin(\gamma\theta_{y2}) e_{\omega_y} - R e_{y2a} - \kappa l_y \sin(\gamma\theta_{y2}) e_{\omega_{yaw}} - u_{y2a} \right) \\
 \dot{e}_{y2b} &= \frac{1}{L} \left(-\kappa \cos(\gamma\theta_{y2}) e_{\omega_y} - R e_{y2b} + \kappa l_y \cos(\gamma\theta_{y2}) e_{\omega_{yaw}} - u_{y2b} \right) \quad (20)
 \end{aligned}$$

□

4. Configuration of LPV System

4.1. LPV System

The varying parameters of the tracking error dynamics are defined as follows:

$$\begin{aligned}
 \mu_1(\theta_{x1}) &= \kappa \sin(\gamma\theta_{x1}) \quad , \quad \mu_2(\theta_{x1}) = \kappa \cos(\gamma\theta_{x1}) \\
 \mu_3(\theta_{x2}) &= \kappa \sin(\gamma\theta_{x2}) \quad , \quad \mu_4(\theta_{x2}) = \kappa \cos(\gamma\theta_{x2}) \\
 \mu_5(\theta_{y1}) &= \kappa \sin(\gamma\theta_{y1}) \quad , \quad \mu_6(\theta_{y1}) = \kappa \cos(\gamma\theta_{y1}) \\
 \mu_7(\theta_{y2}) &= \kappa \sin(\gamma\theta_{y2}) \quad , \quad \mu_8(\theta_{y2}) = \kappa \cos(\gamma\theta_{y2})
 \end{aligned} \quad (21)$$

Given that these are trigonometric functions, the varying parameters are bounded as follows:

$$\underline{\mu}_i = -\kappa \leq \mu_i(\cdot) \leq \bar{\mu}_i = \kappa \quad (22)$$

where $i = 1, 2, 3, 4, 5, 6, 7, 8$.

Therefore, the tracking error dynamics of the Sawyer motor expressed in (19) and (20) can be transformed into an LPV system expressed as follows:

$$\dot{e} = A_e(\mu)e + B_e u \quad (23)$$

The parameter-dependent matrix $A_e(\mu)$ and B_e are defined as follows:

$$A_e(\mu) = \begin{bmatrix} A_{e1,1}(\mu) & A_{e1,2}(\mu) \\ A_{e2,1}(\mu) & A_{e2,2}(\mu) \end{bmatrix}, B_e = \begin{bmatrix} B_{e1,1} \\ B_{e2,1} \end{bmatrix}$$

where $A_{e1,1}(\mu) = \begin{bmatrix} C^{(1)} & C^{(2)} \\ C^{(3)} & C^{(4)} \end{bmatrix}$, $A_{e1,2}(\mu) = \begin{bmatrix} C^{(5)} & C^{(6)} \\ C^{(7)} & C^{(8)} \end{bmatrix}$, $A_{e2,1}(\mu) = \begin{bmatrix} C^{(9)} & C^{(10)} \\ C^{(11)} & C^{(12)} \end{bmatrix}$,
 $A_{e2,2}(\mu) = -\frac{R}{L} \text{diag}[1, 1, 1, 1, 1, 1]$

$$C^{(1)} = \begin{bmatrix} -k_{\theta_x} & 1 & 0 & 0 \\ -\frac{1}{M} & -\frac{(k\omega_x+B_x)}{M} & 0 & 0 \\ 0 & 0 & -k_{\theta_y} & 1 \\ 0 & 0 & -\frac{1}{M} & -\frac{(k\omega_y+B_y)}{M} \end{bmatrix}, C^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu_1(\theta_{x1})}{M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C^{(3)}(\mu) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1(\theta_{x1})}{L} & 0 & 0 \end{bmatrix}, C^{(4)} = \begin{bmatrix} -k_{\theta_{yaw}} & 1 & 0 \\ -\frac{1}{J} & -\frac{(k\omega_{yaw}+B_{yaw})}{J} & -\frac{\mu_1(\theta_{x1})}{J} \\ 0 & \frac{\mu_1(\theta_{x1})l_x}{L} & -\frac{R}{L} \end{bmatrix}$$

$$C^{(5)} = \frac{1}{M} \begin{bmatrix} 0 & 0 & 0 \\ \mu_2(\theta_{x1}) & -\mu_3(\theta_{x2}) & \mu_4(\theta_{x2}) \\ 0 & 0 & 0 \end{bmatrix}, C^{(6)} = 0_{3,4}$$

$$C^{(7)} = \frac{1}{J} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_2(\theta_{x1}) & \mu_3(\theta_{x2}) & -\mu_4(\theta_{x2}) \\ 0 & 0 & 0 \end{bmatrix}$$

$$C^{(8)} = \begin{bmatrix} -\frac{\mu_5(\theta_{y1})}{M} & \frac{\mu_6(\theta_{y1})}{M} & -\frac{\mu_7(\theta_{y2})}{M} & \frac{\mu_8(\theta_{y2})}{M} \\ 0 & 0 & 0 & 0 \\ -\frac{\mu_5(\theta_{y1})}{J} & \frac{\mu_6(\theta_{y1})}{J} & \frac{\mu_7(\theta_{y2})}{J} & -\frac{\mu_8(\theta_{y2})}{J} \\ 0 & 0 & 0 & 0 \end{bmatrix}, C^{(9)} = \frac{1}{L} \begin{bmatrix} 0 & -\mu_2(\theta_{x1}) & 0 \\ 0 & \mu_3(\theta_{x2}) & 0 \\ 0 & -\mu_4(\theta_{x2}) & 0 \end{bmatrix}$$

$$C^{(10)} = \frac{l_x}{L} \begin{bmatrix} 0 & 0 & -\mu_2(\theta_{x1}) & 0 \\ 0 & 0 & -\mu_3(\theta_{x2}) & 0 \\ 0 & 0 & \mu_4(\theta_{x2}) & 0 \end{bmatrix}, C^{(11)} = 0_{4,3}, C^{(12)} = \frac{1}{L} \begin{bmatrix} \mu_5(\theta_{y1}) & 0 & \mu_5(\theta_{y1})l_y & 0 \\ -\mu_6(\theta_{y1}) & 0 & -\mu_6(\theta_{y1})l_y & 0 \\ \mu_7(\theta_{y2}) & 0 & -\mu_7(\theta_{y2})l_y & 0 \\ -\mu_8(\theta_{y2}) & 0 & \mu_8(\theta_{y2})l_y & 0 \end{bmatrix}$$

$$B_{e1,1} = 0_{6,8}, B_{e2,1} = -\frac{1}{L} \text{diag}[1, 1, 1, 1, 1, 1]$$

The varying parameters $\mu_1(\theta_{x1})$, $\mu_2(\theta_{x1})$, $\mu_3(\theta_{x2})$, $\mu_4(\theta_{x2})$, $\mu_5(\theta_{y1})$, $\mu_6(\theta_{y1})$, $\mu_7(\theta_{y2})$, and $\mu_8(\theta_{y2})$ can be continuously updated online during the control process through the measurements of θ_{x1} , θ_{x2} , θ_{y1} , and θ_{y2} . Moreover, the values of these varying parameters are limited by known upper and lower bounds, as expressed in (22). These varying parameters can be described as polytopic:

$$\mu(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}) = V\zeta(\mu) \quad (24)$$

where $\zeta(\mu) = [\zeta_1(\mu) \ \zeta_2(\mu) \ \dots \ \zeta_{15}(\mu) \ \zeta_{16}(\mu)]^T$ is a convex interpolation parameter vector, $V = [\underline{V} \ \overline{V}]$ is a vertex, and $V \in R^{8 \times 16}$ satisfying the following conditions:

$$\underline{V} = \text{diag}[\underline{\mu}_1, \underline{\mu}_2, \underline{\mu}_3, \underline{\mu}_4, \underline{\mu}_5, \underline{\mu}_6, \underline{\mu}_7, \underline{\mu}_8]$$

$$\overline{V} = \text{diag}[\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3, \overline{\mu}_4, \overline{\mu}_5, \overline{\mu}_6, \overline{\mu}_7, \overline{\mu}_8]$$

Therefore, $A_e(\mu)$ determined in the LPV system through (23) can be expressed as follows:

$$A_e(\mu) = \sum_{i=1}^{16} \zeta_i(\mu) A_e^{(i)} \quad (25)$$

where $\xi_i(\mu) \geq 0$, ($i = 1, 2, \dots, 15, 16$), and $\sum_{i=1}^{16} \xi_i(\mu) = 1$.

We define the polytopic decomposition as follows:

$$\begin{aligned} A_e^{(1)} &= A_e(0) + \underline{\mu}_1 \hat{A}_e^{(1)}, & A_e^{(2)} &= A_e(0) + \bar{\mu}_1 \hat{A}_e^{(1)} \\ A_e^{(3)} &= A_e(0) + \underline{\mu}_2 \hat{A}_e^{(2)}, & A_e^{(4)} &= A_e(0) + \bar{\mu}_2 \hat{A}_e^{(2)} \\ &\vdots & & \vdots \\ A_e^{(15)} &= A_e(0) + \underline{\mu}_8 \hat{A}_e^{(8)}, & A_e^{(16)} &= A_e(0) + \bar{\mu}_8 \hat{A}_e^{(8)} \end{aligned} \quad (26)$$

where $A_e(0) = \begin{bmatrix} A_{e1,1}(0) & A_{e1,2}(0) \\ A_{e2,1}(0) & A_{e2,2}(0) \end{bmatrix}$, $A_{e1,2}(0) = 0_{7,7}$, $A_{e2,1}(0) = 0_{7,7}$, and $\hat{A}_e^{(1)}, \hat{A}_e^{(2)} \dots \hat{A}_e^{(7)}, \hat{A}_e^{(8)}$ are real-fixed nodal matrices related to $\mu(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2})$ and extreme values of the varying parameters.

4.2. Vertex Expansion Technique

The convex interpolation parameter vector plays an important role in determining the parameter-dependent matrix $A_e(\mu)$. It is computed based on vertices. Given that the vertex V cannot implement the inverse, we propose the vertex expansion technique through Definition 1:

Definition 1. Consider $\mu(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}) = V\xi(\mu)$ and $\sum_{i=1}^{16} \xi_i(\mu) = 1$, we define $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7$, and β_8 as the constrained polytopic parameters. If $\xi_i \geq 0$, then we have the following expressions:

$$\begin{aligned} \zeta_1(\mu) + \zeta_9(\mu) &= \beta_1, & \zeta_2(\mu) + \zeta_{10}(\mu) &= \beta_2 \\ \zeta_3(\mu) + \zeta_{11}(\mu) &= \beta_3, & \zeta_4(\mu) + \zeta_{12}(\mu) &= \beta_4 \\ \zeta_5(\mu) + \zeta_{13}(\mu) &= \beta_5, & \zeta_6(\mu) + \zeta_{14}(\mu) &= \beta_6 \\ \zeta_7(\mu) + \zeta_{15}(\mu) &= \beta_7, & \zeta_8(\mu) + \zeta_{16}(\mu) &= \beta_8 \end{aligned}$$

such that $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8 = 1$.

According to Definition 1 and the varying parameters limited by known upper and lower bounds, as expressed in (22), $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0.125$. We have extended the V vertex to satisfy the inverse. Thus, the varying parameter vector is expressed as follows:

$$\hat{V} = \begin{bmatrix} 8V \\ N \end{bmatrix} = \begin{bmatrix} 8V & 8\bar{V} \\ I_{8,8} & I_{8,8} \end{bmatrix}, \quad N \in R^{8 \times 16} \quad (27)$$

$$\hat{\mu}(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}) = \begin{bmatrix} \mu(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}) \\ H_{8,1} \end{bmatrix} = \begin{bmatrix} \mu_1(\theta_{x1}) \\ \vdots \\ \mu_8(\theta_{y2}) \\ 0.125 \\ \vdots \\ 0.125 \end{bmatrix} \quad (28)$$

Finally, the convex interpolation parameter vector is computed as follows:

$$\xi(\mu) = \hat{V}^{-1} \hat{\mu}(\theta_{x1}, \theta_{x2}, \theta_{y1}, \theta_{y2}) \quad (29)$$

The vertices must simultaneously comply with the following constraint conditions:

$$\begin{aligned}\mu_1^2 + \mu_2^2 &= \kappa^2 (\sin^2(\gamma\theta_{x1}) + \cos^2(\gamma\theta_{x1})) = \kappa^2 \\ \mu_3^2 + \mu_4^2 &= \kappa^2 (\sin^2(\gamma\theta_{x2}) + \cos^2(\gamma\theta_{x2})) = \kappa^2 \\ \mu_5^2 + \mu_6^2 &= \kappa^2 (\sin^2(\gamma\theta_{y1}) + \cos^2(\gamma\theta_{y1})) = \kappa^2 \\ \mu_7^2 + \mu_8^2 &= \kappa^2 (\sin^2(\gamma\theta_{y2}) + \cos^2(\gamma\theta_{y2})) = \kappa^2\end{aligned}\quad (30)$$

5. Augmented Nonlinear Observer and H_∞ Feedback Control Design Using LPV Synthesis

5.1. Augmented Nonlinear Observer

In the previous section, we assumed that all the state variables are measurable in the error dynamics, LPV system, and the proposed current controller. The measurement of all the state variables is extremely difficult by using sensors because there are several problems, such as cost and space limitations. In addition, to be robust against disturbance in the position control, the load forces and torque as disturbance must be determined. This subsection describes the design of an augmented nonlinear observer by using only position feedback to obtain all the state variables (position, velocity, current) including disturbance. In practice, the load forces and torque vary very slowly; therefore, it can be reasonably supposed that $\dot{d}_x = 0$, $\dot{d}_y = 0$, and, $\dot{d}_{yaw} = 0$. The augmented nonlinear observer is designed as follows:

$$\begin{aligned}\dot{\hat{d}}_x &= l_{dx}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{\theta}}_x &= \hat{\omega}_x + l_{\theta x}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{\omega}}_x &= \frac{-B_x \hat{\omega}_x + \hat{F}_{x1} + \hat{F}_{x2} - \hat{d}_x}{M} + l_{\omega x}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{d}}_y &= l_{dy}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{\theta}}_y &= \hat{\omega}_y + l_{\theta y}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{\omega}}_y &= \frac{-B_y \hat{\omega}_y + \hat{F}_{y1} + \hat{F}_{y2} - \hat{d}_y}{M} + l_{\omega y}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{d}}_{yaw} &= l_{dyaw}(\theta_{yaw} - \hat{\theta}_{yaw}) \\ \dot{\hat{\theta}}_{yaw} &= \hat{\omega}_{yaw} + l_{\theta yaw}(\theta_{yaw} - \hat{\theta}_{yaw}) \\ \dot{\hat{\omega}}_{yaw} &= \frac{-B_{yaw} \hat{\omega}_{yaw} + l_x(\hat{F}_{x1} - \hat{F}_{x2}) + l_y(\hat{F}_{y1} - \hat{F}_{y2}) - \hat{d}_{yaw}}{J} + l_{\omega yaw}(\theta_{yaw} - \hat{\theta}_{yaw}) \\ \dot{\hat{i}}_{x1a} &= \frac{-R\hat{i}_{x1a} + \kappa \sin(\gamma\theta_{x1})(\hat{\omega}_x + l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x1a}}{L} + l_{x1a}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x1b} &= \frac{-R\hat{i}_{x1b} - \kappa \cos(\gamma\theta_{x1})(\hat{\omega}_x + l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x1b}}{L} + l_{x1b}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x2a} &= \frac{-R\hat{i}_{x2a} + \kappa \sin(\gamma\theta_{x2})(\hat{\omega}_x - l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x2a}}{L} + l_{x2a}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x2b} &= \frac{-R\hat{i}_{x2b} - \kappa \cos(\gamma\theta_{x2})(\hat{\omega}_x - l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x2b}}{L} + l_{x2b}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{y1a} &= \frac{-R\hat{i}_{y1a} + \kappa \sin(\gamma\theta_{y1})(\hat{\omega}_y + l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y1a}}{L} + l_{y1a}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y1b} &= \frac{-R\hat{i}_{y1b} - \kappa \cos(\gamma\theta_{y1})(\hat{\omega}_y + l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y1b}}{L} + l_{y1b}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y2a} &= \frac{-R\hat{i}_{y2a} + \kappa \sin(\gamma\theta_{y2})(\hat{\omega}_y - l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y2a}}{L} + l_{y2a}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y2b} &= \frac{-R\hat{i}_{y2b} - \kappa \cos(\gamma\theta_{y2})(\hat{\omega}_y - l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y2b}}{L} + l_{y2b}(\theta_y - \hat{\theta}_y)\end{aligned}\quad (31)$$

where

$$\begin{aligned}\hat{F}_{x1} &= -\kappa \sin(\gamma\theta_{x1})\hat{i}_{x1a} + \kappa \cos(\gamma\theta_{x1})\hat{i}_{x1b}, \hat{F}_{x2} = -\kappa \sin(\gamma\theta_{x2})\hat{i}_{x2a} + \kappa \cos(\gamma\theta_{x2})\hat{i}_{x2b} \\ \hat{F}_{y1} &= -\kappa \sin(\gamma\theta_{y1})\hat{i}_{y1a} + \kappa \cos(\gamma\theta_{y1})\hat{i}_{y1b}, \hat{F}_{y2} = -\kappa \sin(\gamma\theta_{y2})\hat{i}_{y2a} + \kappa \cos(\gamma\theta_{y2})\hat{i}_{y2b}\end{aligned}\quad (33)$$

Let us define the vector X_a , the estimation vector $\hat{X}_a$, and the estimation error vector $\tilde{X}_a$ of the augmented state variables as follows:

$$\begin{aligned} X_a &= \begin{bmatrix} X_{mecha}^T & X_{eleca}^T \end{bmatrix}^T \\ X_{mecha} &= \begin{bmatrix} d_x & \theta_x & \omega_x & d_y & \theta_y & \omega_y & d_{yaw} & \theta_{yaw} & \omega_{yaw} \end{bmatrix}^T \\ X_{eleca} &= \begin{bmatrix} i_{x1a} & i_{x1b} & i_{x2a} & i_{x2b} & i_{y1a} & i_{y1b} & i_{y2a} & i_{y2b} \end{bmatrix}^T \end{aligned} \quad (34)$$

and

$$\begin{aligned} \hat{X}_a &= \begin{bmatrix} \hat{X}_{mecha}^T & \hat{X}_{eleca}^T \end{bmatrix}^T \\ \hat{X}_{mecha} &= \begin{bmatrix} \hat{d}_x & \hat{\theta}_x & \hat{\omega}_x & \hat{d}_y & \hat{\theta}_y & \hat{\omega}_y & \hat{d}_{yaw} & \hat{\theta}_{yaw} & \hat{\omega}_{yaw} \end{bmatrix}^T \\ \hat{X}_{eleca} &= \begin{bmatrix} \hat{i}_{x1a} & \hat{i}_{x1b} & \hat{i}_{x2a} & \hat{i}_{x2b} & \hat{i}_{y1a} & \hat{i}_{y1b} & \hat{i}_{y2a} & \hat{i}_{y2b} \end{bmatrix}^T \\ \tilde{X}_a &= \begin{bmatrix} \tilde{X}_{mecha}^T & \tilde{X}_{eleca}^T \end{bmatrix}^T \\ \tilde{X}_{mecha} &= \begin{bmatrix} \tilde{d}_x & \tilde{\theta}_x & \tilde{\omega}_x & \tilde{d}_y & \tilde{\theta}_y & \tilde{\omega}_y & \tilde{d}_{yaw} & \tilde{\theta}_{yaw} & \tilde{\omega}_{yaw} \end{bmatrix}^T \\ \tilde{X}_{eleca} &= \begin{bmatrix} \tilde{i}_{x1a} & \tilde{i}_{x1b} & \tilde{i}_{x2a} & \tilde{i}_{x2b} & \tilde{i}_{y1a} & \tilde{i}_{y1b} & \tilde{i}_{y2a} & \tilde{i}_{y2b} \end{bmatrix}^T \end{aligned} \quad (35)$$

where

$$\begin{aligned} \tilde{d}_x &= d_x - \hat{d}_x, \quad \tilde{\theta}_x = \theta_x - \hat{\theta}_x, \quad \tilde{\omega}_x = \omega_x - \hat{\omega}_x \\ \tilde{d}_y &= d_y - \hat{d}_y, \quad \tilde{\theta}_y = \theta_y - \hat{\theta}_y, \quad \tilde{\omega}_y = \omega_y - \hat{\omega}_y \\ \tilde{d}_{yaw} &= d_{yaw} - \hat{d}_{yaw}, \quad \tilde{\theta}_{yaw} = \theta_{yaw} - \hat{\theta}_{yaw}, \quad \tilde{\omega}_{yaw} = \omega_{yaw} - \hat{\omega}_{yaw} \\ \tilde{i}_{x1a} &= i_{x1a} - \hat{i}_{x1a}, \quad \tilde{i}_{x1b} = i_{x1b} - \hat{i}_{x1b} \\ \tilde{i}_{x2a} &= i_{x2a} - \hat{i}_{x2a}, \quad \tilde{i}_{x2b} = i_{x2b} - \hat{i}_{x2b} \\ \tilde{i}_{y1a} &= i_{y1a} - \hat{i}_{y1a}, \quad \tilde{i}_{y1b} = i_{y1b} - \hat{i}_{y1b} \\ \tilde{i}_{y2a} &= i_{y2a} - \hat{i}_{y2a}, \quad \tilde{i}_{y2b} = i_{y2b} - \hat{i}_{y2b} \end{aligned} \quad (36)$$

and

$$\begin{aligned} \tilde{d} &= \begin{bmatrix} \tilde{d}_x & \tilde{d}_y & \tilde{d}_{yaw} \end{bmatrix}^T \\ \tilde{F}_{x1} &= -\kappa \sin(\gamma\theta_{x1})\tilde{i}_{x1a} + \kappa \cos(\gamma\theta_{x1})\tilde{i}_{x1b}, \quad \tilde{F}_{x2} = -\kappa \sin(\gamma\theta_{x2})\tilde{i}_{x2a} + \kappa \cos(\gamma\theta_{x2})\tilde{i}_{x2b} \\ \tilde{F}_{y1} &= -\kappa \sin(\gamma\theta_{y1})\tilde{i}_{y1a} + \kappa \cos(\gamma\theta_{y1})\tilde{i}_{y1b}, \quad \tilde{F}_{y2} = -\kappa \sin(\gamma\theta_{y2})\tilde{i}_{y2a} + \kappa \cos(\gamma\theta_{y2})\tilde{i}_{y2b} \end{aligned} \quad (37)$$

The estimation error dynamics of the augmented state variables can be given as follows:

$$\dot{\tilde{X}}_a = (A_a - LC_a)\tilde{X}_a + B_a\tilde{d} \quad (38)$$

where $A_a = \begin{bmatrix} A_{a1,1} & A_{a1,2} \\ A_{a2,1} & A_{a2,2} \end{bmatrix}$, $A_{a1,1} = \text{diag}[A_{a1,1x}, A_{a1,1y}, A_{a1,1yaw}]$, and

$$\begin{aligned}
A_{a1,1x} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_x}{M} \end{bmatrix}, \quad A_{a1,1y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_y}{M} \end{bmatrix}, \quad A_{a1,1yaw} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{B_{yaw}}{M} \end{bmatrix} \\
A_{a1,2} &= \begin{bmatrix} R^{(a1)} & R^{(a2)} & R^{(a3)} & R^{(a4)} & R^{(a5)} & R^{(a6)} \end{bmatrix}^T \\
R^{(a1)} &= 0_{2,8}, R^{(a2)} = \frac{1}{M} \begin{bmatrix} -S_{x1} & C_{x1} & -S_{x2} & C_{x2} & 0_{1,4} \end{bmatrix}, R^{(a3)} = 0_{2,8} \\
R^{(a4)} &= \frac{1}{M} \begin{bmatrix} 0_{1,4} & -S_{y1} & C_{y1} & -S_{y2} & C_{y2} \end{bmatrix}, R^{(a5)} = 0_{2,8} \\
R^{(a6)} &= \frac{1}{J} \begin{bmatrix} -l_x S_{x1} & l_x C_{x1} & l_x S_{x2} & -l_x C_{x2} & -l_y S_{y1} & l_y C_{y1} & l_y S_{y2} & -l_y C_{y2} \end{bmatrix} \\
A_{a2,1} &= \begin{bmatrix} C^{(a1)} & C^{(a2)} & C^{(a3)} & C^{(a4)} & C^{(a5)} & C^{(a6)} \end{bmatrix} \\
C^{(a1)} &= 0_{8,2}, C^{(a2)} = \frac{1}{L} \begin{bmatrix} S_{x1} & -C_{x1} & S_{x2} & -C_{x2} & 0_{1,4} \end{bmatrix}^T, C^{(a3)} = 0_{8,2} \\
C^{(a4)} &= \frac{1}{L} \begin{bmatrix} 0_{1,4} & S_{y1} & -C_{y1} & S_{y2} & -C_{y2} \end{bmatrix}^T, C^{(a5)} = 0_{8,2} \\
C^{(a6)} &= \frac{C_{\theta yaw}}{L} \begin{bmatrix} l_x S_{x1} & -l_x C_{x1} & -l_x S_{x2} & l_x C_{x2} & l_y S_{y1} & -l_y C_{y1} & -l_y S_{y2} & l_y C_{y2} \end{bmatrix}^T \\
A_{a2,2} &= -\frac{R}{L} \text{diag}[1, 1, 1, 1, 1, 1, 1, 1], L = \text{diag} \begin{bmatrix} L^{(1)} & L^{(2)} \end{bmatrix} \\
L^{(1)} &= \begin{bmatrix} l_{dx} & l_{\theta x} & l_{\omega x} & l_{dy} & l_{\theta y} & l_{\omega y} & l_{dyaw} & l_{\theta yaw} & l_{\omega yaw} \end{bmatrix} \\
L^{(2)} &= \begin{bmatrix} l_{x1a} & l_{x1b} & l_{x2a} & l_{x2b} & l_{y1a} & l_{y1b} & l_{y2a} & l_{y2b} \end{bmatrix} \\
C_a &= \begin{bmatrix} C^{(a7)} & C^{(a8)} & C^{(a9)} & C^{(a10)} & C^{(a11)} & C^{(a12)} & C^{(a13)} \end{bmatrix} \\
C^{(a7)} &= 0_{17,1}, C^{(a8)} = \begin{bmatrix} 1 & 1 & 1 & 0_{1,6} & 1 & 1 & 1 & 1 & 0_{1,4} \end{bmatrix}^T \\
C^{(a9)} &= 0_{17,2}, C^{(a10)} = \begin{bmatrix} 0_{1,3} & 1 & 1 & 1 & 0_{1,7} & 1 & 1 & 1 & 1 \end{bmatrix}^T \\
C^{(a11)} &= 0_{17,2}, C^{(a12)} = \begin{bmatrix} 0_{1,6} & 1 & 1 & 1 & 0_{1,8} \end{bmatrix}^T, C^{(a13)} = 0_{17,9} \\
B_a &= \begin{bmatrix} C^{(a14)} & C^{(a15)} & C^{(a16)} \end{bmatrix}, C^{(a14)} = -\frac{1}{M} \begin{bmatrix} 0_{1,2} & 1 & 0_{1,14} \end{bmatrix}^T \\
C^{(a15)} &= -\frac{1}{M} \begin{bmatrix} 0_{1,5} & 1 & 0_{1,11} \end{bmatrix}^T, C^{(a16)} = -\frac{1}{J} \begin{bmatrix} 0_{1,8} & 1 & 0_{1,8} \end{bmatrix}^T \\
S_{x1} &= \kappa \sin(\gamma \theta_{x1}), C_{x1} = \kappa \cos(\gamma \theta_{x1}), S_{x2} = \kappa \sin(\gamma \theta_{x2}), C_{x2} = \kappa \cos(\gamma \theta_{x2}) \\
S_{y1} &= \kappa \sin(\gamma \theta_{y1}), C_{y1} = \kappa \cos(\gamma \theta_{y1}), S_{y2} = \kappa \sin(\gamma \theta_{y2}), C_{y2} = \kappa \cos(\gamma \theta_{y2}) \\
C_{\theta yaw} &= \cos(\theta_{yaw})
\end{aligned}$$

Theorem 2. Consider the estimation error dynamics of the augmented state variables (38). If an observer gain matrix L can be chosen such that:

$$\sigma < \frac{1}{2\lambda_{\max}(P_a)}$$

where $(A_a - LC_a)^T P_a + P_a (A_a - LC_a) = -I$ and $\lambda_{\max}(P_a)$ is the maximum eigenvalue of P_a . Then, $\tilde{X}_a$ exponentially converges to zero.

Proof of Theorem 2. Let us consider the Lyapunov function candidate:

$$V_a = \tilde{X}_a^T P_a \tilde{X}_a \quad (39)$$

The derivative of V_a with respect to time is expressed as follows:

$$\begin{aligned}
\dot{V}_a &= \tilde{X}_a^T \left[(A_a - LC_a)^T P_a + P_a (A_a - LC_a) \right] \tilde{X}_a + 2\tilde{X}_a^T P_a B_a \tilde{d} \\
&= -\tilde{X}_a^T \tilde{X}_a + 2\tilde{X}_a^T P_a B_a \tilde{d} \\
&\leq -\|\tilde{X}_a\|_2^2 + 2\sigma \|P_a\|_2 \|\tilde{X}_a\|_2^2 \\
&\leq -\|\tilde{X}_a\|_2^2 + 2\sigma \lambda_{\max}(P_a) \|\tilde{X}_a\|_2^2
\end{aligned}$$

Thus, if $\sigma < \frac{1}{2\lambda_{\max}(P_a)}$, then $\tilde{X}_a$ exponentially converges to zero [21]. $\square$

Remark 1: Since the proposed method guarantees the exponential stability of the zero equilibrium point of (39), the proposed method guarantees the robustness against the

parameter uncertainties if the parameter uncertainties can be regarded as the vanishing perturbation term [21]. However, if the parameter uncertainties are in the form of the nonvanishing perturbation term, the proposed method guarantees only the boundedness of the tracking error and the estimation error [21].

5.2. LPV H_∞ State Feedback Control and Closed-Loop Stability Analysis

In the previous subsection, we discussed the augmented nonlinear observer design to obtain all the state variables (position, velocity, current) including the load forces and torque as disturbance. In this subsection, we present the design of the pre-defined control inputs u_{x1a} , u_{x1b} , u_{x2a} , u_{x2b} , u_{y1a} , u_{y1b} , u_{y2a} , and u_{y2b} in the context of a linear system using H_∞ optimization. The estimated state variables are used instead of the measured signals so that the control performance is not affected by the measurement noise. Let us define the tracking-error state vectors of the augmented state variables as follows:

$$\begin{aligned} e_a &= \begin{bmatrix} e_{mecha}^T & e_{eleca}^T \end{bmatrix}^T \\ e_{mecha} &= \begin{bmatrix} -d_x & e_{\theta_x} & e_{\omega_x} & -d_y & e_{\theta_y} & e_{\omega_y} & -d_{yaw} & e_{\theta_{yaw}} & e_{\omega_{yaw}} \end{bmatrix}^T \\ e_{eleca} &= \begin{bmatrix} e_{x1a} & e_{x1b} & e_{x2a} & e_{x2b} & e_{y1a} & e_{y1b} & e_{y2a} & e_{y2b} \end{bmatrix}^T \end{aligned} \quad (40)$$

From (23), the tracking error dynamics in the LPV system becomes:

$$\begin{aligned} \dot{e}_a &= A_{ea}(\mu)e_a + B_{ea}u + B_w\tilde{d} \\ z &= C_1e_a + D_{12}u \end{aligned} \quad (41)$$

where $z \in R^{17}$ is the objective function signal, $\tilde{d} = \begin{bmatrix} \tilde{d}_x & \tilde{d}_y & \tilde{d}_{yaw} \end{bmatrix}^T$ is as the exogenous disturbance signal, $B_w^T B_w \in R^{17 \times 17}$ is the covariance matrix, and $C_1 \in R^{17 \times 17}$ and $D_{12} \in R^{17 \times 8}$ are the weighting matrices of the tracking errors and control inputs, respectively:

$$\begin{aligned} A_{ea} &= \begin{bmatrix} A_{ea1,1} & A_{ea1,2} \\ A_{ea2,1} & A_{ea2,2} \end{bmatrix} \\ A_{ea1,1} &= \text{diag}[A_{ea1,1x}, A_{ea1,1y}, A_{ea1,1yaw}] \\ A_{ea1,1x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -k_{\theta_x} & 1 \\ 0 & -\frac{1}{M} & -\frac{(k_{\omega_x} + B_x)}{M} \end{bmatrix}, A_{ea1,1y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -k_{\theta_y} & 1 \\ 0 & -\frac{1}{M} & -\frac{(k_{\omega_y} + B_y)}{M} \end{bmatrix} \\ A_{ea1,1yaw} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -k_{\theta_{yaw}} & 1 \\ 0 & -\frac{1}{J} & -\frac{(k_{\omega_{yaw}} + B_{yaw})}{J} \end{bmatrix} \\ A_{ea1,2} &= \begin{bmatrix} R^{(ea1)} & R^{(ea2)} & R^{(ea3)} & R^{(ea4)} & R^{(ea5)} & R^{(ea6)} \end{bmatrix}^T \\ R^{(ea1)} &= 0_{2,8}, R^{(ea2)} = \frac{1}{M} \begin{bmatrix} -\mu_1(\theta_{x1}) & \mu_2(\theta_{x1}) & -\mu_3(\theta_{x2}) & \mu_4(\theta_{x2}) & 0_{1,4} \end{bmatrix} \\ R^{(ea3)} &= 0_{2,8}, R^{(ea4)} = \frac{1}{M} \begin{bmatrix} 0_{1,4} & -\mu_5(\theta_{y1}) & \mu_6(\theta_{y1}) & -\mu_7(\theta_{y2}) & \mu_8(\theta_{y2}) \end{bmatrix} \\ R^{(ea5)} &= 0_{2,8}, R^{(ea6)} = \frac{1}{J} \begin{bmatrix} R^{(ea61)} & R^{(ea62)} \end{bmatrix} \\ R^{(ea61)} &= \begin{bmatrix} -\mu_1(\theta_{x1}) & \mu_2(\theta_{x1}) & \mu_3(\theta_{x2}) & -\mu_4(\theta_{x2}) \end{bmatrix} \\ R^{(ea62)} &= \begin{bmatrix} -\mu_5(\theta_{y1}) & \mu_6(\theta_{y1}) & \mu_7(\theta_{y2}) & -\mu_8(\theta_{y2}) \end{bmatrix} \\ A_{ea2,1} &= \begin{bmatrix} C^{(ea1)} & C^{(ea2)} & C^{(ea3)} & C^{(ea4)} & C^{(ea5)} & C^{(ea6)} \end{bmatrix} \\ C^{(ea1)} &= 0_{8,2}, C^{(ea2)} = \frac{1}{L} \begin{bmatrix} \mu_1(\theta_{x1}) & -\mu_2(\theta_{x1}) & \mu_3(\theta_{x2}) & -\mu_4(\theta_{x2}) & 0_{1,4} \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned}
C^{(ea3)} &= 0_{8,2}, C^{(ea4)} = \frac{1}{L} \begin{bmatrix} 0_{1,4} & \mu_5(\theta_{y1}) & -\mu_6(\theta_{y1}) & \mu_7(\theta_{y2}) & -\mu_8(\theta_{y2}) \end{bmatrix}^T \\
C^{(ea5)} &= 0_{8,2}, C^{(ea6)} = \frac{1}{L} \begin{bmatrix} C^{(ea61)} & C^{(ea62)} \end{bmatrix}^T \\
C^{(ea61)} &= l_x \begin{bmatrix} \mu_1(\theta_{x1}) & -\mu_2(\theta_{x1}) & -\mu_3(\theta_{x2}) & \mu_4(\theta_{x2}) \end{bmatrix} \\
C^{(ea62)} &= l_y \begin{bmatrix} \mu_5(\theta_{y1}) & -\mu_6(\theta_{y1}) & -\mu_7(\theta_{y2}) & \mu_8(\theta_{y2}) \end{bmatrix} \\
A_{ea2,2} &= -\frac{R}{L} \text{diag}[1, 1, 1, 1, 1, 1, 1, 1] \\
B_{ea} &= \begin{bmatrix} R^{(ea7)} & R^{(ea8)} \end{bmatrix}^T, R^{(ea7)} = 0_{9,8}, R^{(ea8)} = -\frac{1}{L} \text{diag}[1, 1, 1, 1, 1, 1, 1, 1]
\end{aligned}$$

The LPV H_∞ state feedback control law becomes as follows:

$$u = K_\infty \hat{e}_a \quad (42)$$

where $K_\infty \in R^{8 \times 17}$ is the LPV gain scheduling.

The closed-loop system dynamics of the proposed control method is illustrated in Figure 2. The state vectors are defined as follows:

$$\begin{aligned}
\theta^d &= \begin{bmatrix} \theta_x^d & \theta_y^d & \theta_{yaw}^d \end{bmatrix}^T, \theta = \begin{bmatrix} \theta_x & \theta_y & \theta_{yaw} \end{bmatrix}^T, \omega = \begin{bmatrix} \omega_x & \omega_y & \omega_{yaw} \end{bmatrix}^T \\
\hat{\theta} &= \begin{bmatrix} \hat{\theta}_x & \hat{\theta}_y & \hat{\theta}_{yaw} \end{bmatrix}^T, i^d = \begin{bmatrix} i_{x1a}^d & i_{x1b}^d & i_{x2a}^d & i_{x2b}^d & i_{y1a}^d & i_{y1b}^d & i_{y2a}^d & i_{y2b}^d \end{bmatrix}^T \\
\hat{\omega} &= \begin{bmatrix} \hat{\omega}_x & \hat{\omega}_y & \hat{\omega}_{yaw} \end{bmatrix}^T, i = \begin{bmatrix} i_{x1a} & i_{x1b} & i_{x2a} & i_{x2b} & i_{y1a} & i_{y1b} & i_{y2a} & i_{y2b} \end{bmatrix}^T \\
v &= \begin{bmatrix} v_{x1a} & v_{x1b} & v_{x2a} & v_{x2b} & v_{y1a} & v_{y1b} & v_{y2a} & v_{y2b} \end{bmatrix}^T
\end{aligned}$$

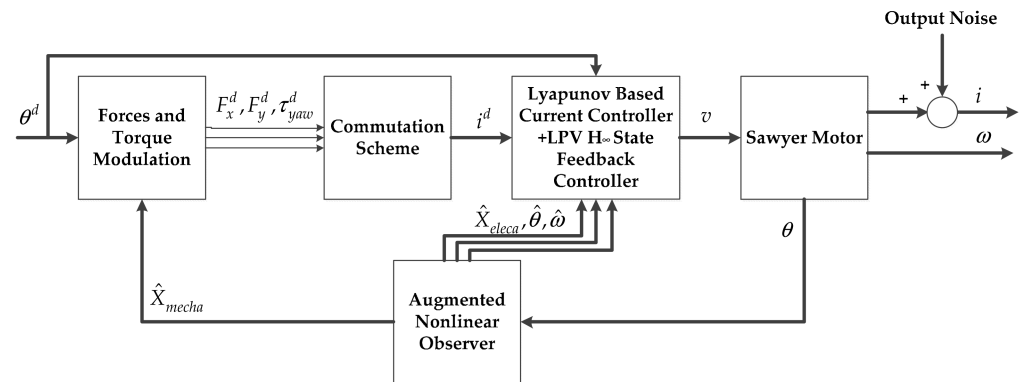


Figure 2. Structure of the proposed control method.

The stability of the closed-loop system dynamics consists of the estimation error dynamics of the augmented state variables (38) and tracking errors dynamics (41) as follows:

$$\underbrace{\begin{bmatrix} \dot{e}_a \\ \dot{\tilde{X}}_a \end{bmatrix}}_{\dot{X}_{cl}} = \underbrace{\begin{bmatrix} A_{ea}(\mu) + & B_{ea}K_\infty \\ 0 & A_a - LC_a \end{bmatrix}}_{A_{cl}} \underbrace{\begin{bmatrix} e_a \\ \tilde{X}_a \end{bmatrix}}_{X_{cl}} + \underbrace{\begin{bmatrix} B_w \\ B_a \end{bmatrix}}_{B_{cl}} \tilde{d} \quad (43)$$

Theorem 3. Consider the closed-loop system dynamics (43). If the LPV gain scheduling K_∞ and observer gain matrix L can be chosen such that:

$$\sigma < \frac{1}{2\lambda_{\max}(P_{Acl})}$$

where $A_{cl}^T P_{Acl} + P_{Acl} A_{cl} = -I$ and $\lambda_{\max}(P_{Acl})$ is the maximum eigenvalue of P_{Acl} . Then, X_{cl} exponentially converges to zero.

Proof of Theorem 3. Let us consider the Lyapunov function candidate:

$$V_{Acl} = X_{cl}^T P_{Acl} X_{cl} \quad (44)$$

The derivative of V_{Acl} with respect to time is expressed as follows:

$$\begin{aligned} \dot{V}_{Acl} &= X_{cl}^T [A_{cl}^T P_{Acl} + P_{Acl} A_{cl}] X_{cl} + 2X_{cl}^T P_{Acl} B_{cl} \tilde{d} \\ &= -X_{cl}^T X_{cl} + 2X_{cl}^T P_{Acl} B_{cl} \tilde{d} \\ &\leq -\|X_{cl}\|_2^2 + 2\sigma \|P_{Acl}\|_2 \|X_{cl}\|_2^2 \\ &\leq -\|X_{cl}\|_2^2 + 2\sigma \lambda_{\max}(P_{Acl}) \|X_{cl}\|_2^2 \end{aligned}$$

Thus, if $\sigma < \frac{1}{2\lambda_{\max}(P_{Acl})}$, then X_{cl} exponentially converges to zero [21]. $\square$

To determine the LPV gain scheduling K_∞ , we apply the H_∞ control in the LMI technique with Assumption 1 in the optimal control design [22]:

Assumption 1.

- (A1) $(A_{ea}(\mu), B_{ea})$ is stable.
- (A2) $D_{12}^T D_{12}$ is invertible.
- (A3) $D_{12}^T C_1 = 0$.
- (A4) $(C_1, A_{ea}(\mu))$ has no unobservable modes on the imaginary axis.

Note that the H_∞ norm of the LPV control system is represented as follows:

$$\|T_{z\tilde{d}}\|_\infty = \sup \frac{\|z\|_2}{\|\tilde{d}\|_2} < \lambda, \quad \forall \tilde{d} \neq 0 \quad (45)$$

The LPV H_∞ state feedback control will determine the bound λ on the H_∞ norm of the LPV control system with the linear matrix inequality (LMI) condition. The following statements are equivalent.

- There exists $u = K_\infty \hat{e}_a$ such that $\|T_{z\tilde{d}}\|_\infty < \lambda$.
- There exists $\Gamma_\infty > 0$ such that:

$$\begin{bmatrix} (A_{ea}(\mu) + B_{ea}K_\infty)\Gamma_\infty & & & \\ +\Gamma_\infty(A_{ea}(\mu) + B_{ea}K_\infty)^T & B_w & \Gamma_\infty(C_1 + D_{12}K_\infty)^T & \\ & B_w & -I & 0 \\ (C_1 + D_{12}K_\infty)\Gamma_\infty & 0 & -\lambda^2 I & \end{bmatrix} < 0 \quad (46)$$

Given that we design the H_∞ state feedback control based on the LPV system, $A_{ea}(\mu)$ is interpolated by the polytopic decomposition $A_{ea}^{(i)}$. Therefore, the linear matrix inequality given by (46) is redefined as follows:

$$\begin{bmatrix} (A_{ea}^{(i)} + B_{ea}K_\infty^{(i)})\Gamma_\infty & & & \\ +\Gamma_\infty(A_{ea}^{(i)} + B_{ea}K_\infty^{(i)})^T & B_w & \Gamma_\infty(C_1 + D_{12}K_\infty^{(i)})^T & \\ & B_w & -I & 0 \\ (C_1 + D_{12}K_\infty^{(i)})\Gamma_\infty & 0 & -\lambda^2 I & \end{bmatrix} < 0 \quad (47)$$

Then, we define a new variable as $\Omega_{\infty}^{(i)} = K_{\infty}^{(i)} \Gamma_{\infty}$, so the LPV gain scheduling of the state feedback control is recovered by $K_{\infty}^{(i)} = \Omega_{\infty}^{(i)} \Gamma_{\infty}^{-1}$ with the following LMI condition:

$$\begin{bmatrix} A_{ea}^{(i)} \Gamma_{\infty} + \Gamma_{\infty} (A_{ea}^{(i)})^T & B_w & \Gamma_{\infty} C_1^T + (\Omega_{\infty}^{(i)})^T D_{12}^T \\ + B_{ea} \Omega_{\infty}^{(i)} + (\Omega_{\infty}^{(i)})^T B_{ea}^T & & \\ B_w & -I & 0 \\ C_1 \Gamma_{\infty} + D_{12} \Omega_{\infty}^{(i)} & 0 & -\lambda^2 I \end{bmatrix} < 0 \quad (48)$$

Finally, the LPV H_{∞} state feedback control law is expressed as follows:

$$u = \sum_{i=1}^{16} \xi_i K_{\infty}^{(i)} \hat{e}_a \quad (49)$$

6. Simulation Results

We utilized the Sawyer motor parameters, control gains, and augmented nonlinear observer gains listed in Table 1. The desired position profiles were those of the ramp function as shown in Figure 3, and the yaw desired profile was set to 0 rad.

Table 1. Sawyer motor parameters, control gains, and augmented nonlinear observer gains.

Parameters	Values	Parameters	Values
J	$4 \times 10^{-3} \text{ kg.m}^2$	M	1.8 kg
κ	17N/A	l_x, l_y	0.0485 m
p	$1.016 \times 10^{-3} \text{ m}$	B_x, B_y, B_{yaw}	$1 \times 10^{-5} \text{ N.m.s/rad}$
R	2Ω	λ	1×10^5
L	$7 \times 10^{-4} \text{ H}$	$k_{\theta_x}, k_{\theta_y}, k_{\theta_{yaw}}$	40
γ	$2 \times \pi \times p$	$k_{\omega_x}, k_{\omega_y}, k_{\omega_{yaw}}$	1
l_{dx}, l_{dy}	-1×10^7	$l_{\omega_x}, l_{\omega_y}$	3.89×10^{-4}
l_{dyaw}	-3×10^3	$l_{\omega_{yaw}}$	0.175
$l_{\theta_x}, l_{\theta_y}$	1×10^3	$l_{x1a}, l_{x1b}, l_{x2a}, l_{x2b}$	0
$l_{\theta_{yaw}}$	1×10^2	$l_{y1a}, l_{y1b}, l_{y2a}, l_{y2b}$	0

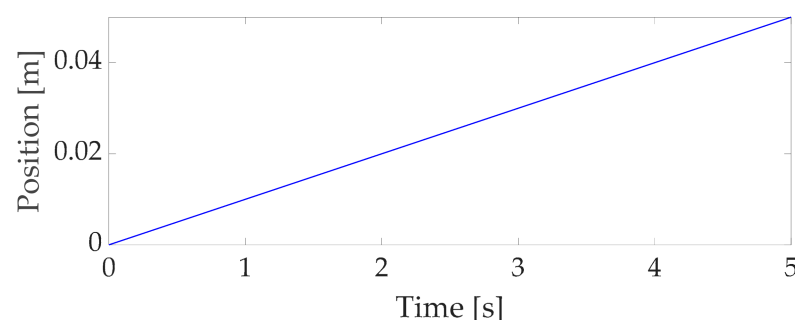
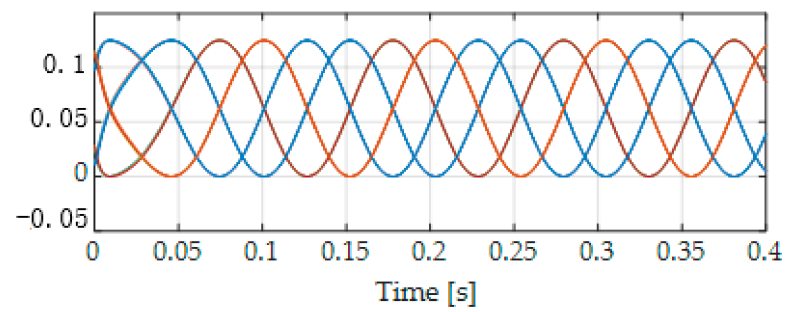
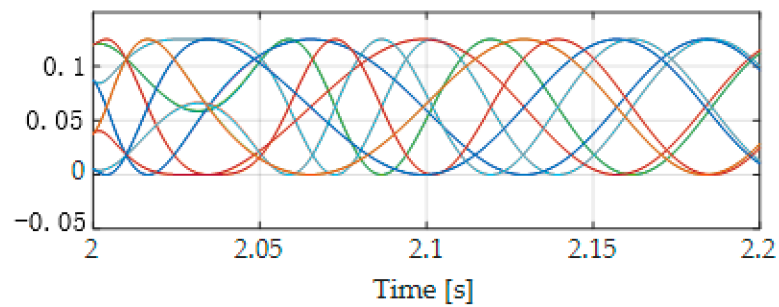


Figure 3. X-axis and Y-axis desired position profiles.

The results of the 16 interpolation parameters are shown in Figure 4 to satisfy the constraint conditions related to the proposed vertex expansion technique in Section 4. The maximum value of each interpolation parameter is 0.125, in which the influence of the load forces and torque as exogenous disturbance of the X-axis, Y-axis, and yaw at 2 s is shown in Figure 5. From these results, we confirm that the proposed LPV control system performs well.

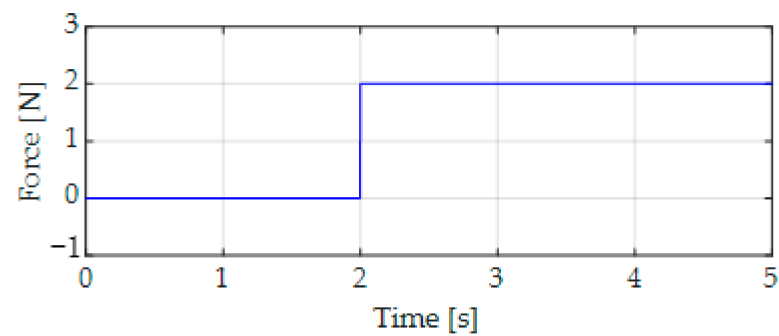


(a)

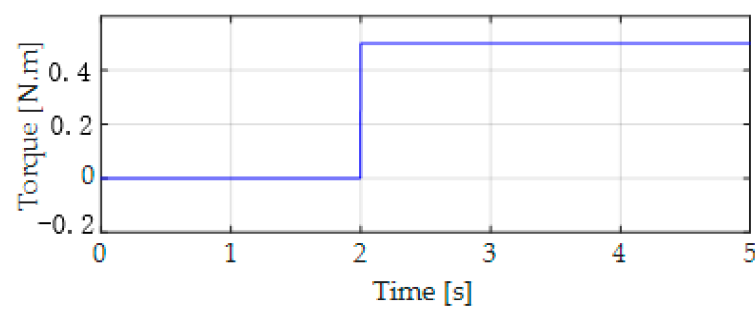


(b)

Figure 4. Sixteen interpolation parameters of the LPV control system. (a) Zoom-in X-axis at the start time of 16 interpolation parameters. (b) Zoom-in X-axis at 2 s of 16 interpolation parameters.



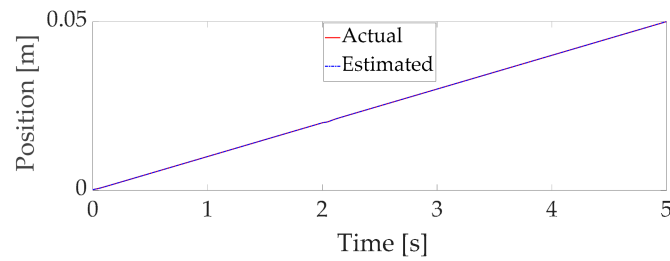
(a)



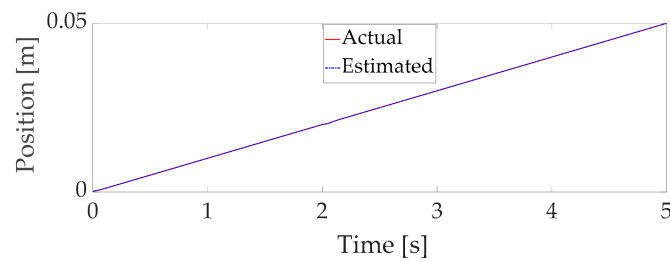
(b)

Figure 5. Load forces and load torque. (a) Load forces d_x and d_y (b) Load torque d_{yaw} .

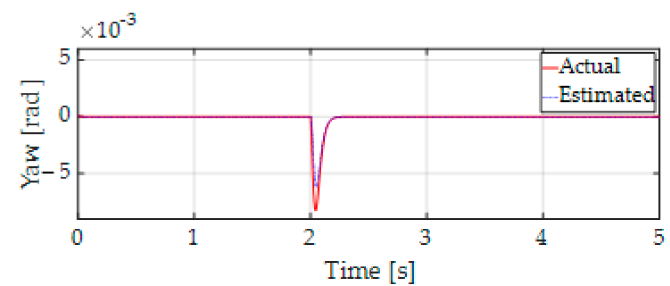
The estimation performance of the proposed ANOB of positions of the X-axis, Y-axis, yaw rotation, the load forces and torque, and phase currents is confirmed as shown in Figure 6. In addition, the estimation error performance is also represented in Figure 7. We verify that the load forces and torque and the state variables were well estimated by applying the proposed ANOB in the presence of 5% output noise of the current.



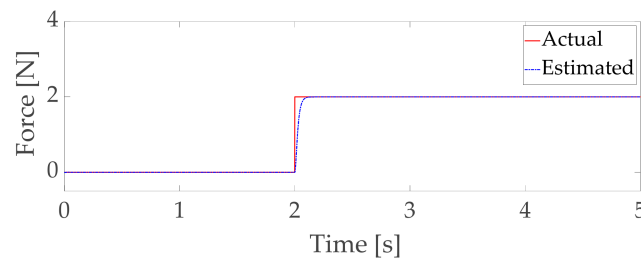
(a)



(b)

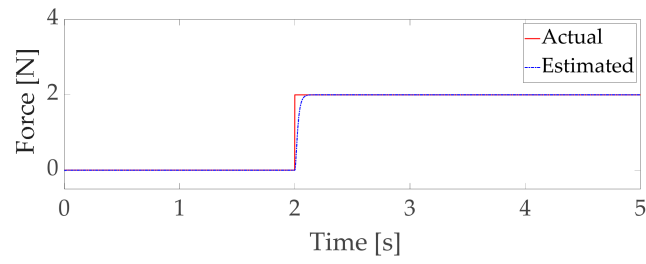


(c)

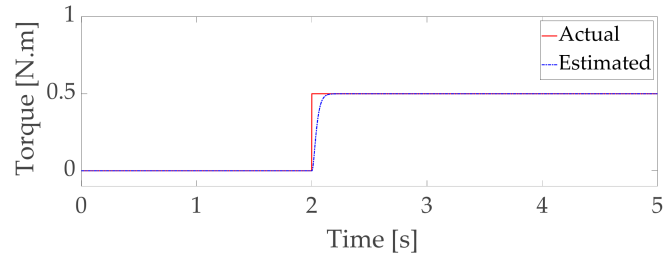


(d)

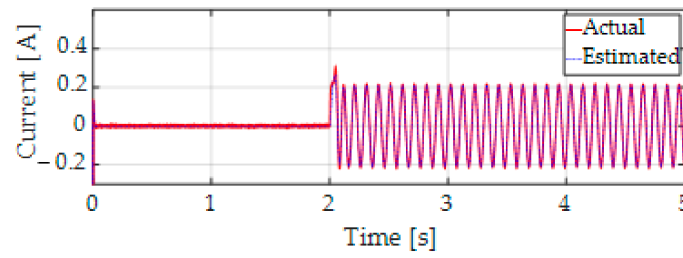
Figure 6. Cont.



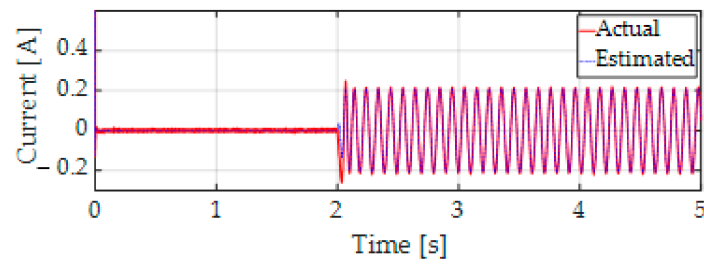
(e)



(f)

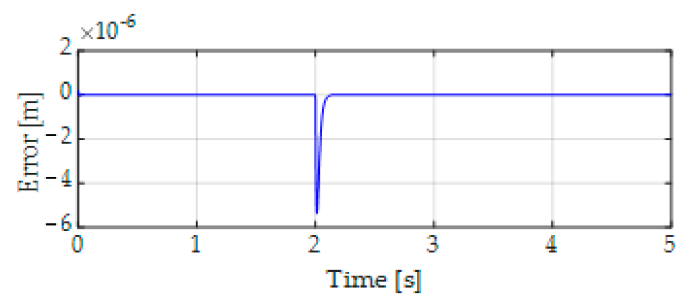


(g)

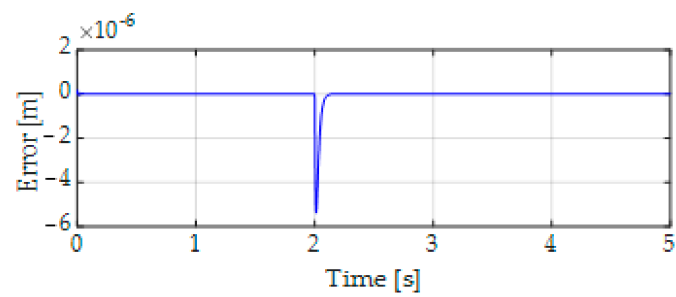


(h)

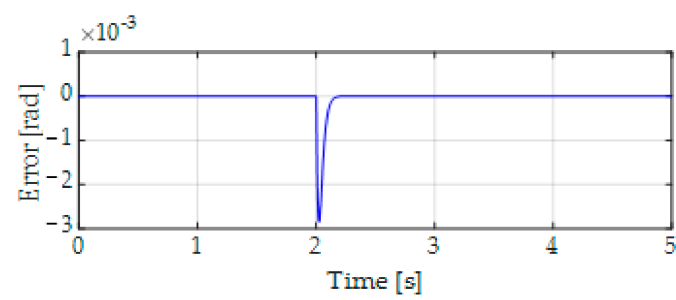
Figure 6. Estimation performance of the proposed ANOB. (a) Estimation performance of θ_x and $\hat{\theta}_x$. (b) Estimation performance of θ_y and $\hat{\theta}_y$. (c) Estimation performance of θ_{yaw} and $\hat{\theta}_{yaw}$. (d) Estimation performance of d_x and $\hat{d}_x$. (e) Estimation performance of d_y and $\hat{d}_y$. (f) Estimation performance of d_{yaw} and $\hat{d}_{yaw}$. (g) Estimation performance of i_{x1a} and $\hat{i}_{x1a}$. (h) Estimation performance of i_{x1b} and $\hat{i}_{x1b}$.



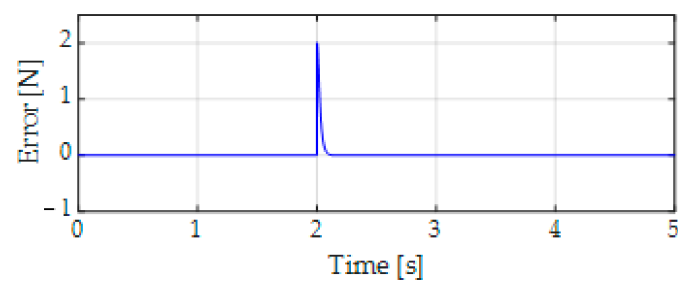
(a)



(b)

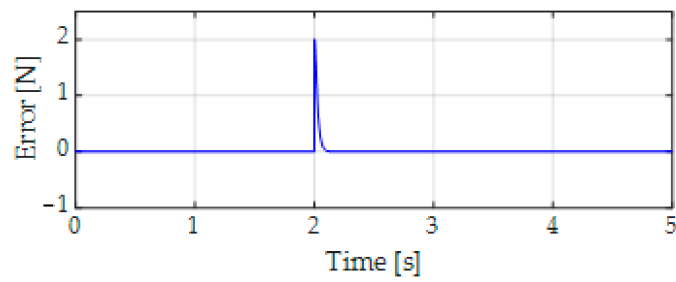


(c)

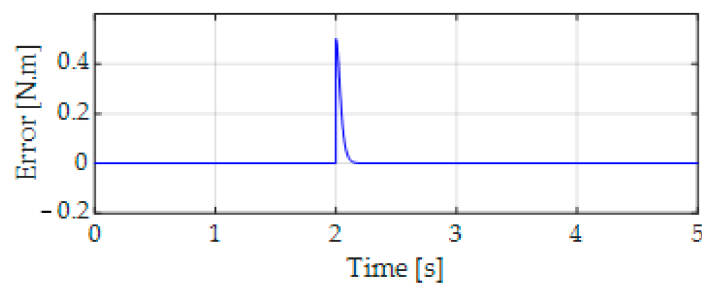


(d)

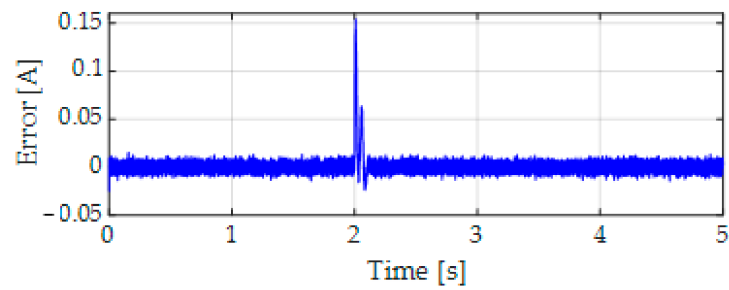
Figure 7. Cont.



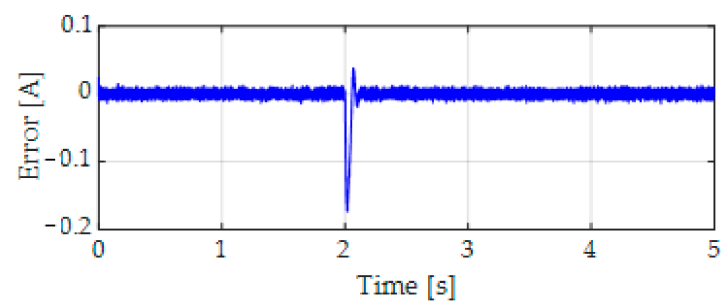
(e)



(f)



(g)



(h)

Figure 7. Estimation error performance of the proposed ANOB. (a) Estimation error performance of $\tilde{\theta}_x$. (b) Estimation error performance of $\tilde{\theta}_y$. (c) Estimation error performance of $\tilde{\theta}_{yaw}$. (d) Estimation error performance of $\tilde{d}_x$. (e) Estimation error performance of $\tilde{d}_y$. (f) Estimation error performance of $\tilde{d}_{yaw}$. (g) Estimation error performance of $\tilde{i}_{x1a}$. (h) Estimation error performance of $\tilde{i}_{x1b}$.

To evaluate the effectiveness of the proposed control method in the influence of the load forces and torque as exogenous disturbance, output noise as well as the robustness to uncertain parameters by 5% change of the nominal parameters including inertia J and coefficients of friction B_x , B_y , and B_{yaw} of the motor, we represented the simulation results in the following two cases:

Case 1: Conventional PID control.

Case 2: LPV H_∞ control with an ANOB.

The position and yaw tracking errors of the two cases are shown in Figure 8. Note that the conventional PID control in Case 1 had a large overshoot during the acceleration and the deceleration periods as well as violent oscillations of yaw motion. The effect of controller gains in Case 1 was high for tracking error dynamics, which means a long settling time in the transient response for the position and yaw tracking error control.

In Case 2, the position and yaw tracking errors were improved, and the oscillation of yaw motion was considerably reduced in the proposed control method because the LPV gain scheduling was controlled to regulate the tracking errors. The convergences of position and yaw tracking errors are zero. Figure 8 also shows that the yaw motion is well regulated in the influence of the load forces and torque as exogenous disturbance at 2 s and uncertain parameters because the effectiveness of the H_∞ control was guaranteed by using the H_∞ norm in (45) to be robust against disturbance. We confirm that the proposed control method improves the tracking error performance and rejects disturbance better than the conventional PID control method.

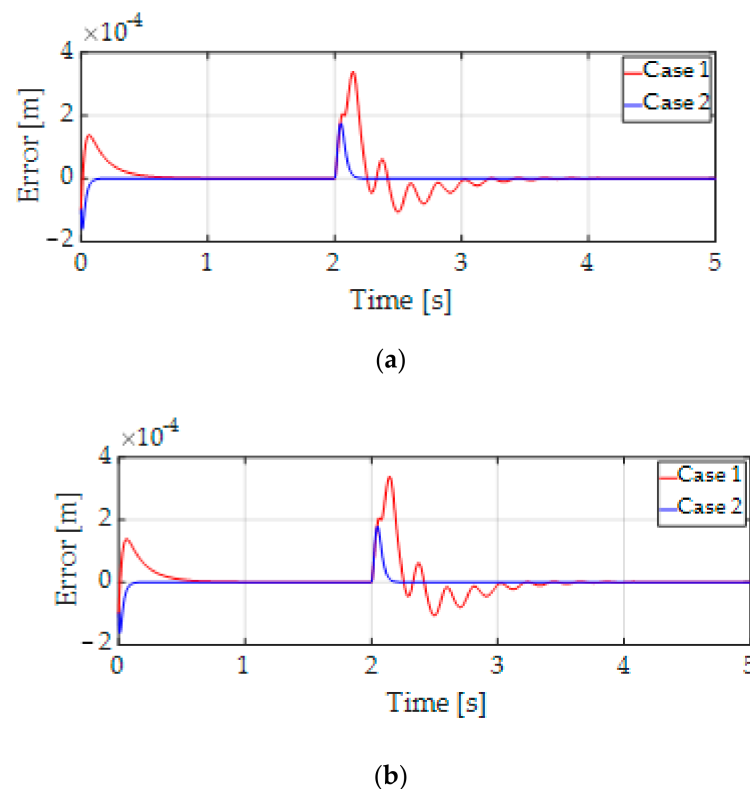
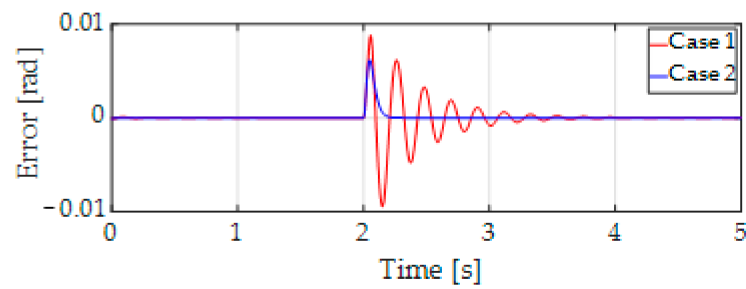
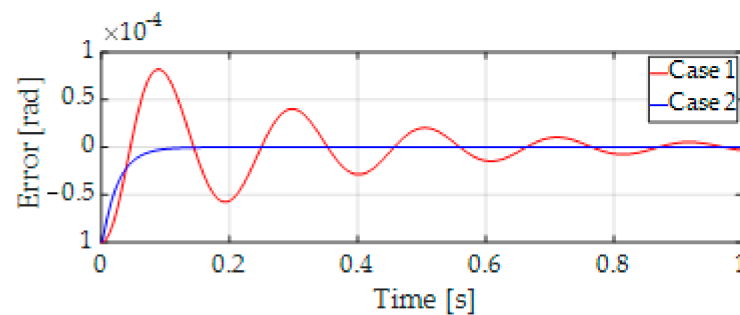


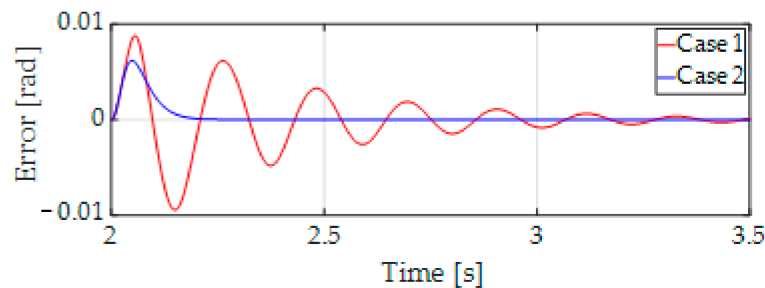
Figure 8. Cont.



(c)



(d)



(e)

Figure 8. Position and yaw tracking errors. (a) X-axis position tracking error of e_{θ_x} . (b) Y-axis position tracking error of e_{θ_y} . (c) Yaw tracking error of $e_{\theta_{yaw}}$. (d) Zoom-in X-axis at the start time of $e_{\theta_{yaw}}$. (e) Zoom-in X-axis at 2 s of $e_{\theta_{yaw}}$.

7. Conclusions

This study proposes LPV H_∞ control with an augmented nonlinear observer for the Sawyer motors. The proposed control method consists of the forces and torque modulation scheme, the augmented nonlinear observer, and the Lyapunov-based current controller with the LPV H_∞ state feedback controller to reduce the tracking error dynamics. The forces and torque modulation scheme is developed to guarantee the stability of the position and yaw tracking errors for the mechanical tracking error dynamics and to generate the desired phase currents. The augmented nonlinear observer is designed to estimate all the state variables including disturbance using only the position feedback. The Lyapunov-based current controller with an LPV H_∞ state feedback controller is proposed to guarantee the stability of phase current tracking errors for the electrical tracking error dynamics. The effectiveness of the proposed control method was verified through simulation results and

compared with the conventional PID control method, showing improved tracking error performance and suitable disturbance rejection.

One possible future research direction is to consider polytopic representation of the LPV system using the tensor product approach to derive a polytopic convex model with various convex hull representations [23–25].

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Abbreviations

LPV	Linear parameter varying
ANOB	Augmented nonlinear observer
LMI	Linear matrix inequality
PID	Proportional integral derivative
DOB	Disturbance observer
EMF	Electromotive force

References

1. Krishnamurthy, P.; Khorrami, F. Robust adaptive control of Sawyer motors without current measurements. *IEEE/ASME Trans. Mechatron.* **2004**, *9*, 689–696. [\[CrossRef\]](#)
2. Krishnamurthy, P.; Khorrami, F.; Ng, T.L.; Cherepinsky, I. Control design and implementation for Sawyer motors used in manufacturing systems. *IEEE Trans. Control Syst. Technol.* **2011**, *19*, 1467–1478. [\[CrossRef\]](#)
3. Pan, J.; Cheung, N.C.; Yang, J. High-precision position control of a novel planar switched reluctance motor. *IEEE Trans. Ind. Electron.* **2005**, *52*, 1644–1652. [\[CrossRef\]](#)
4. Nguyen, V.H.; Kim, W.J. Design and control of a compact light-weight planar positioner moving over a concentrated-field magnet matrix. *IEEE/ASME Trans. Mechatron.* **2013**, *18*, 1090–1099. [\[CrossRef\]](#)
5. Hu, C.; Wang, Z.; Zhu, Y.; Zhang, M.; Liu, H. Performance-oriented precision LARC tracking motion control of a magnetically levitated planar motor with comparative experiments. *IEEE Trans. Ind. Electron.* **2016**, *63*, 5763–5773. [\[CrossRef\]](#)
6. Huang, S.D.; Chen, L.; Cao, G.Z.; Wu, C.; Xu, J.; He, Z. Predictive position control of planar motors using trajectory gradient soft constraint with attenuation coefficients in the weighting matrix. *IEEE Trans. Ind. Electron.* **2021**, *68*, 821–837. [\[CrossRef\]](#)
7. Huang, S.D.; Cao, G.Z.; Xu, J.; Cui, Y.; Wu, C.; He, J. Predictive position control of long-stroke planar motors for high-precision positioning applications. *IEEE Trans. Ind. Electron.* **2021**, *68*, 796–811. [\[CrossRef\]](#)
8. Apkarian, P.; Gahinet, P.; Becker, G. Self-scheduled H_∞ control of linear parameter-varying systems: A design example. *Automatica* **1995**, *31*, 1251–1261. [\[CrossRef\]](#)
9. Packard, A. Gain scheduling via linear fractional transformations. *Syst. Control Lett.* **1994**, *22*, 79–92. [\[CrossRef\]](#)
10. Shamma, S. Analysis and Design of Gain Scheduled Control Systems. Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA, USA, 1988.
11. Kim, K.S.; Rew, K.H.; Kim, S. Disturbance observer for estimating higher order disturbances in time series expansion. *IEEE Trans. Automat. Contr.* **2010**, *55*, 1905–1911.
12. Ishikawa, J.; Tomizuka, M. Pivot friction compensation using an accelerometer and a disturbance observer for hard disk drives. *IEEE/ASME Trans. Mechatron.* **1998**, *3*, 194–201. [\[CrossRef\]](#)
13. Kim, K.H.; Youn, M.J. A nonlinear speed control for a PM synchronous motor using a simple disturbance estimation technique. *IEEE Trans. Ind. Electron.* **2002**, *49*, 524–535.
14. Liu, H.; Li, S. Speed control for PMSM servo system using predictive functional control and extended state observer. *IEEE Trans. Ind. Electron.* **2012**, *59*, 1171–1183. [\[CrossRef\]](#)
15. Cho, K.; Kim, J.; Choi, S.B.; Oh, S. A high-precision motion control based on a periodic adaptive disturbance observer in a PMLSM. *IEEE/ASME Trans. Mechatron.* **2015**, *20*, 2158–2171. [\[CrossRef\]](#)

16. Lee, S.H.; Kang, H.J.; Chung, C.C. Robust fast seek control of a servo track writer using a state space disturbance observer. *IEEE Trans. Control Syst. Technol.* **2012**, *20*, 346–355. [\[CrossRef\]](#)
17. Kang, H.J.; Kim, K.S.; Lee, S.H.; Chung, C.C. Bias compensation for fast servo track writer seek Control. *IEEE Trans. Magn.* **2011**, *47*, 1937–1943. [\[CrossRef\]](#)
18. Delibas, B.; Koc, B. A method to realize low velocity movability and eliminate friction induced noise in piezoelectric ultrasonic motors. *IEEE/ASME Trans. Mechatron.* **2020**, *25*, 2677–2687. [\[CrossRef\]](#)
19. Ding, S.; Chen, W.H.; Mei, K.; Murray-Smith, D.J. Disturbance observer design for nonlinear systems represented by input-output models. *IEEE Trans. Ind. Electron.* **2020**, *67*, 1222–1232. [\[CrossRef\]](#)
20. Boyd, S.; Vandenberghe, L. *Convex Optimization*; Cambridge University Press: Cambridge, UK, 2003.
21. Khalil, H.K. *Nonlinear Systems*, 3rd ed.; Prentice-Hall: Englewood Cliffs, NJ, USA, 2002.
22. Lee, Y.W.; Lee, S.H.; Chung, C.C. LPV H_∞ Control with disturbance estimation for permanent magnet synchronous motors. *IEEE Trans. Ind. Electron.* **2018**, *65*, 488–497. [\[CrossRef\]](#)
23. Baranyi, P.; Yam, Y.; Varlaki, P. *Tensor Product Model Transformation in Polytopic Model-Based Control*; CRC Press: Boca Raton, FL, USA, 2013.
24. Szollosi, A.; Baranyi, P. Influence of the tensor product model representation of qLPV models on the feasibility of linear matrix inequality. *Asian J. Control* **2016**, *18*, 1328–1342. [\[CrossRef\]](#)
25. Takarics, B.; Vanek, B. Robust control design for the FLEXOP demonstrator aircraft via tensor product models. *Asian J. Control* **2021**, *23*, 1290–1300. [\[CrossRef\]](#)