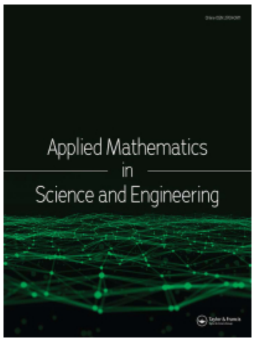

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Some well-known inequalities of Ostrowski like for Caputo derivatives

Yonghong Liu[a], Ghulam Farid[b], Jongsuk Ro[c,d], Mawahib Elamin[e] and Sayed Abdel-Khalek[f]

[a]School of Computer Science, Chengdu University, Chengdu, People's Republic of China; [b]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan; [c]School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul, Republic of Korea; [d]Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul, Republic of Korea; [e]Department of Mathematics, College of Science, Qassim University, Buraydah, Saudi Arabia; [f]Department of Mathematics and Statistics, College of Science, Taif University, Taif, Saudi Arabia

ABSTRACT
This paper aims to provide new versions of some known inequalities by applying Caputo fractional derivatives. Ostrowski, Hermite-Hadamard and Ostrowski-Grüss-type inequalities are given. Generalized conditions of existing inequalities are analysed to get new inequalities. Special cases are discussed to prove the general nature of former inequalities.

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1. Introduction

Mathematics is a universal language and every real-world problem can be described in mathematical models. Inequalities play a vital role in solving such models and proving the existence and uniqueness of their solutions. In the literature, a lot of celebrated new and classical inequalities can be studied. It is a common practice that the authors investigate generalizations, refinements and extensions of already known inequalities. We are interested in analysing inequalities proved in [1–3], by applying Caputo fractional derivatives. In the following, well-known Ostrowski, Grüss, Ostrowski-Grüss and Hermite-Hadamard inequalities are given.

Theorem 1.1 **([1]):** *Let $\Psi : I \longrightarrow \mathbb{R}$ be a differentiable mapping in I^o, the interior of I and $x_1, x_2 \in I^o$, $x_1 < x_2$. If $|\Psi'(t)| \le \mathcal{M}$ for all $t \in [x_1, x_2]$. Then for $\sigma \in [x_1, x_2]$, one*

CONTACT Jongsuk Ro jongsukro@gmail.com School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea; Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea

have:

$$\left|\Psi(\sigma)-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Psi(\varphi)\,\mathrm{d}\varphi\right|\leq\left[\frac{1}{4}+\frac{\left(\sigma-\frac{x_1+x_2}{2}\right)^2}{(x_2-x_1)^2}\right](x_2-x_1)\mathcal{M}. \tag{1}$$

Inequality (1) provides the estimation of the value of function at an arbitrary point to the integral mean of that function over a given interval. It deals with bounded differentiable functions and finds applications in numerical computations of integrals, see [4, 5] and references therein. The Grüss inequality is stated as follows:

Theorem 1.2 ([1]): *Let $\Psi,\ \Phi:[x_1,x_2]\longrightarrow\mathbb{R}$, be two integrable functions. If $m\leq\Psi(t)\leq M$ and $n\leq\Phi(t)\leq N$ for all $t\in[x_1,x_2]$. Then one have:*

$$\left|\int_{x_1}^{x_2}\Psi(\sigma)\Phi(\sigma)\,\mathrm{d}\sigma-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Psi(\sigma)\,\mathrm{d}\sigma\cdot\int_{x_1}^{x_2}\Phi(\sigma)\,\mathrm{d}\sigma\right|\leq\frac{1}{4}(M-m)(N-n). \tag{2}$$

Inequality (2) provides estimations of integral mean of the product of two functions to the product of their integral means and deals with the bounded functions. By applying Grüss inequality, an Ostrowski-type inequality was established by Dragomir in [3], as stated in the following theorem. In the literature, it is well known as the first Ostrowski-Grüss inequality.

Theorem 1.3 ([3]): *Let $\Psi:I\longrightarrow\mathbb{R}$ be a differentiable mapping in I^o. If $m\leq\Psi'(t)\leq M$, then for $\sigma\in[x_1,x_2]$ one have:*

$$\begin{aligned}&\left|\Psi(\sigma)-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Psi(\varphi)\,\mathrm{d}\varphi-\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1}\left(\sigma-\frac{x_1+x_2}{2}\right)\right|\\&\leq\frac{1}{4}(x_2-x_1)(M-m).\end{aligned} \tag{3}$$

Many authors received results directly connected with Ostrowski and Ostrowski-Grüss inequalities, see [5–14]. Next, we give the well-known Hermite-Hadamard inequality.

Theorem 1.4: *Let $\Psi:[x_1,x_2]\to\mathbb{R}$ be a convex function. Then one have:*

$$\Psi\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\Psi(\varphi)\,\mathrm{d}\varphi\leq\frac{\Psi(x_1)+\Psi(x_2)}{2}. \tag{4}$$

The above inequalities were studied extensively, and a lot of works exist on mathematical inequalities. For Ostrowski-type inequalities, one can go through the papers [5–9]. For Grüss and Ostrowski-Grüss-type inequalities, one can go through the papers [10–13]. For Hadamard-type inequalities, one can see [15–20].

The aim of this paper is to derive the aforementioned inequalities for Caputo fractional derivatives, which are defined in the following.

Definition 1.1 ([21]): Let $\Phi \in AC^n[a,b]$ and $n = [\Re(\tau)] + 1$ ($\Re(\tau)$ denote the real part of τ). Then Caputo fractional derivatives of order $\tau \in \mathbb{C}$, $\Re(\tau) > 0$ of Φ are defined as follows:

$$^{C}D^{\tau}_{a+}\Phi(x) = \frac{1}{\Gamma(n-\tau)} \int_a^x \frac{\Phi^{(n)}(t)}{(x-t)^{\tau-n+1}}\,\mathrm{d}t, \quad x > a, \tag{5}$$

and

$$^{C}D^{\tau}_{b-}\Phi(x) = \frac{(-1)^n}{\Gamma(n-\tau)} \int_x^b \frac{\Phi^{(n)}(t)}{(t-x)^{\tau-n+1}}\,\mathrm{d}t, \quad x < b. \tag{6}$$

If $\tau = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $(^{C}D^{\tau}_{a+}\Phi)(x)$ coincides with $\Phi^{(n)}(x)$, whereas $(^{C}D^{\tau}_{b-}\Phi)(x)$ coincides with $\Phi^{(n)}(x)$ for a constant multiplier $(-1)^n$. In particular, we have

$$(^{C}D^{0}_{a+}\Phi)(x) = (^{C}D^{0}_{b-}\Phi)(x) = \Phi(x) \tag{7}$$

where $n = 1$ and $\tau = 0$.

The Caputo derivative introduces memory effects or non-locality into diffusion and reaction-diffusion models, offering a means to describe anomalous diffusions and systems with long-range interactions. In the Keller-Segel model, replacing the classical time derivatives with Caputo (fractional) derivatives allows for the modelling of anomalous chemotaxis, where the movement of organisms and the diffusion of the chemical signal display non-local or history-dependent behaviour. This modification results in a more generalized and adaptable framework, capable of describing more complex biological and physical phenomena beyond the classical Keller-Segel theory, see [22–25].

Farid in [14] proved the Ostrowski inequality without considering the piece-wise function and the corresponding Motgomery identity, also he proved some related well-known inequalities in alternative ways. We follow the same method in proving Ostrowski and Ostrowski-Grüss-type inequalities for Caputo fractional derivatives.

2. Ostrowski-type inequalities for Caputo fractional derivatives

First, we give the following Ostrowski-type inequality for Caputo fractional derivatives.

Theorem 2.1: *Let $\Psi \in C^{(n+1)}[x_1, x_2]$ with $|\Psi^{(n+1)}(\varphi)| \leq \eta(\varphi)$ for all $\varphi \in [x_1, x_2]$, where $\eta \in L[x_1, x_2]$. Then for $\sigma \in [x_1, x_2]$, the inequality given below holds for Caputo derivatives:*

$$\begin{aligned}&\Big|(\sigma - x_1)^{n-\mu}\Psi^{(n)}(\sigma) + (x_2 - \sigma)^{n-\nu}\Psi^{(n)}(\sigma) - \Big(\Gamma(n-\mu+1)^{C}D^{\mu}_{\sigma-}\Phi(x_1)\\ &+\Gamma(n-\nu+1)^{C}D^{\nu}_{\sigma+}\Phi(x_2)\Big)\Big| \leq \int_{x_1}^{\sigma} (\sigma - x_1)^{n-\mu}\eta(\varphi)\,\mathrm{d}\varphi + \int_{\sigma}^{x_2} (x_2 - \sigma)^{n-\nu}\eta(\varphi)\,\mathrm{d}\varphi,\end{aligned} \tag{8}$$

where μ, $\nu > 0$.

Proof: Let $\varphi \in [x_1, \sigma]$. Then $\Psi^{(n+1)}$ according to the statement must satisfy the following inequality for $\mu, \nu > 0$:

$$\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\mu} \Psi^{(n+1)}(\varphi)\, \mathrm{d}\varphi + \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\mu} \eta(\varphi)\, \mathrm{d}\varphi \geq 0.$$

Since $(\sigma - x_1)^{n-\mu} \geq (\varphi - x_1)^{n-\mu}$ for $\mu, \nu > 0$, integrating by parts and using this fact we have

$$\begin{aligned}
&(\sigma - x_1)^{n-\mu} \Psi^{(n)}(\sigma) - (n-\mu) \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\mu-1} \Psi^{(n)}(\varphi)\, \mathrm{d}\varphi \\
&\geq - \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\mu} \eta(\varphi)\, \mathrm{d}\varphi \geq - \int_{x_1}^{\sigma} (\sigma - x_1)^{n-\mu} \eta(\varphi)\, \mathrm{d}\varphi. \qquad (9)
\end{aligned}$$

Using the definition of Caputo fractional derivatives in the above inequality, one can obtain

$$(\sigma - x_1)^{n-\mu} \Psi^{(n)}(\sigma) - \Gamma(n-\mu+1)^{C} D_{\sigma-}^{\mu} \Phi(x_1) \geq - \int_{x_1}^{\sigma} (\sigma - x_1)^{n-\mu} \eta(\varphi)\, \mathrm{d}\varphi. \quad (10)$$

Similarly, for $\varphi \in [\sigma, x_2]$ and $\mu, \nu > 0$, we have

$$\int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\nu} \eta(\varphi)\, \mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\nu} \Psi^{(n+1)}(\varphi)\, \mathrm{d}\varphi \geq 0.$$

Integrating by parts using $x_2 - \sigma \geq x_2 - \varphi$, we have

$$\begin{aligned}
&(x_2 - \sigma)^{n-\nu} \Psi^{(n)}(\sigma) - (n-\nu) \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\nu-1} \Psi^{(n)}(\varphi)\, \mathrm{d}\varphi \\
&\geq - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\nu} \eta(\varphi)\, \mathrm{d}\varphi \geq - \int_{\sigma}^{x_2} (x_2 - \sigma)^{n-\nu} \eta(\varphi)\, \mathrm{d}\varphi. \qquad (11)
\end{aligned}$$

Using the definition of Caputo fractional derivatives in the above inequality, one can have

$$(x_2 - \sigma)^{n-\nu} \Psi^{(n)}(\sigma) - \Gamma(n-\nu+1)^{C} D_{\sigma+}^{\nu} \Phi(x_2) \geq - \int_{\sigma}^{x_2} (x_2 - \sigma)^{n-\nu} \eta(\varphi)\, \mathrm{d}\varphi. \quad (12)$$

Summing inequalities (10) and (11), one can get:

$$\begin{aligned}
&(\sigma - x_1)^{n-\mu} \Psi^{(n)}(\sigma) + (x_2 - \sigma)^{n-\nu} \Psi^{(n)}(\sigma) - \Big(\Gamma(n-\mu+1)^{C} D_{\sigma-}^{\mu} \Phi(x_1) \\
&+ \Gamma(n-\nu+1)^{C} D_{\sigma+}^{\nu} \Phi(x_2)\Big) \\
&\geq - \left(\int_{x_1}^{\sigma} (\sigma - x_1)^{n-\mu} \eta(\varphi)\, \mathrm{d}\varphi + \int_{\sigma}^{x_2} (x_2 - \sigma)^{n-\nu} \eta(\varphi)\, \mathrm{d}\varphi \right). \qquad (13)
\end{aligned}$$

Similarly, using non-negativity of terms $(\varphi - x_1)^{n-\mu}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))$, and $(x_2 - \varphi)^{n-\nu}(\eta(\varphi) + \Psi^{(n+1)}(\varphi))$ for $\mu, \nu > 0$, one can get:

$$\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\mu} (\eta(\varphi) - \Psi^{(n+1)}(\varphi))\, \mathrm{d}\varphi \geq 0, \qquad (14)$$

$$\int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\nu} (\eta(\varphi) + \Psi^{(n+1)}(\varphi))\, \mathrm{d}\varphi \geq 0. \qquad (15)$$

Integrating by parts, the above integrals provide the following inequality:

$$(\sigma - x_1)^{n-\mu}\Psi^{(n)}(\sigma) + (x_2 - \sigma)^{n-\nu}\Psi^{(n)}(\sigma) - \Big(\Gamma(n-\mu+1)^C D^{\mu}_{\sigma-}\Phi(x_1) + \Gamma(n-\nu+1)^C D^{\nu}_{\sigma+}\Phi(x_2)\Big) \le \int_{x_1}^{\sigma} (\sigma - x_1)^{n-\mu}\eta(\varphi)\,\mathrm{d}\varphi + \int_{\sigma}^{x_2} (x_2 - \sigma)^{n-\nu}\eta(\varphi)\,\mathrm{d}\varphi. \tag{16}$$

Required inequality (8) can be obtained from (13) and (16). ■

Applications of Theorem 2.1 are given in the following results.

Corollary 2.1: *Under the assumptions of Theorem 2.1, the following Hermite-Hadamard-type inequality holds for Caputo fractional derivatives:*

$$\left|\frac{\Psi^{(n)}(x_1) + \Psi^{(n)}(x_2)}{2} - \frac{\Gamma(n-\mu+1)}{2(x_2 - x_1)^{\mu}}\left[{}^C D^{\mu}_{\sigma-}\Psi(x_1) + {}^C D^{\mu}_{\sigma+}\Phi(x_2)\right]\right| \le \int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi.$$

Proof: By fixing $\sigma = x_2$ in (10) and $\sigma = x_1$ in (12) respectively, the upcoming inequalities are yielded

$$\Psi^{(n)}(x_2) - \frac{\Gamma(n-\mu+1)^C D^{\mu}_{\sigma-}\Psi(x_1)}{(x_2 - x_1)^{n-\mu}} \ge -\int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi,$$

$$\Psi^{(n)}(x_1) - \frac{\Gamma(n-\nu+1)^C D^{\mu}_{\sigma+}\Psi(x_2)}{(x_2 - x_1)^{n-\nu}} \ge -\int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi.$$

The above two inequalities yield the incoming inequality

$$\frac{\Psi^{(n)}(x_1) + \Psi^{(n)}(x_2)}{2} - \left(\frac{\Gamma(n-\mu+1)^C D^{\mu}_{\sigma-}\Psi(x_1)}{2(x_2 - x_1)^{n-\mu}} + \frac{\Gamma(n-\nu+1)^C D^{\nu}_{\sigma+}\Psi(x_2)}{2(x_2 - x_1)^{n-\nu}}\right) \ge -\int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi.$$

Similarly, by fixing $\sigma = x_2$ in the resulting inequality of (14) and $\sigma = x_1$ in the resulting inequality of (15), respectively, and one can add the upcoming inequality

$$\frac{\Psi^{(n)}(x_1) + \Psi^{(n)}(x_2)}{2} - \left(\frac{\Gamma(n-\mu+1)^C D^{\mu}_{\sigma-}\Psi(x_1)}{2(x_2 - x_1)^{n-\mu}} + \frac{\Gamma(n-\nu+1)^C D^{\nu}_{\sigma+}\Psi(x_2)}{2(x_2 - x_1)^{n-\nu}}\right) \le \int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi.$$

From the above two inequalities, one can obtain the following modulus inequality:

$$\left|\frac{\Psi^{(n)}(x_1) + \Psi^{(n)}(x_2)}{2} - \left(\frac{\Gamma(n-\mu+1)^C D^{\mu}_{u_2-}\Psi(x_1)}{2(x_2 - x_1)^{n-\mu}} + \frac{\Gamma(n-\nu+1)^C D^{\nu}_{u_1+}\Psi(x_2)}{2(x_2 - x_1)^{n-\nu}}\right)\right| \le \int_{x_1}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi.$$

By putting $\mu = \nu$, from the above inequality, the required result is obtained. ■

Another variant of Hermite-Hadamard type inequality for Caputo fractional derivatives is given in the following.

Corollary 2.2: *Under the assumptions of Theorem* 2.1, *the following inequality holds:*

$$\left|\frac{\Gamma(n-\mu+1)}{2^{\mu-n+1}(x_2-x_1)^{(n-\mu)}}\left({}^{C}D^{\mu}_{\frac{x_1+x_2}{2}-}\Phi(x_1)+{}^{C}D^{\mu}_{\frac{x_1+x_2}{2}+}\Phi(x_2)\right)-\Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)\right|$$
$$\leq 2^{\mu-n}(x_2-x_1)^{n-\mu}\int_{x_1}^{x_2}\eta(\varphi)\,\mathrm{d}\varphi.$$

Proof: By setting $\sigma = \frac{x_1+x_2}{2}$ in (8), we get

$$\left|\left(2^{\mu-n}(x_2-x_1)^{n-\mu}+2^{\nu-n}(x_2-x_1)^{n-\nu}\right)\Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)\right.$$
$$\left.-\left(\Gamma(n-\mu+1){}^{C}D^{\mu}_{\frac{x_1+x_2}{2}-}\Phi(x_1)+\Gamma(n-\nu+1){}^{C}D^{\nu}_{\frac{x_1+x_2}{2}+}\Phi(x_2)\right)\right|$$
$$\leq 2^{\mu-n}(x_2-x_1)^{n-\mu}\int_{x_1}^{\frac{x_1+x_2}{2}}\eta(\varphi)\,\mathrm{d}\varphi+2^{\nu-n}(x_2-x_1)^{n-\nu}\int_{\frac{x_1+x_2}{2}}^{x_2}\eta(\varphi)\,\mathrm{d}\varphi. \tag{17}$$

Now, we fix $\mu = \nu$ in (17), to get the required inequality. The proof is completed. ■

In the following theorem, we establish Hermite-Hadamard-type inequality for Caputo fractional derivatives with bounds in the form of Caputo derivatives:

Theorem 2.2: *Let* $\Psi \in C^{(n+1)}[x_1,x_2]$ *with* $|\Psi^{(n+1)}(\varphi)| \leq \zeta(\varphi)$ *for all* $\varphi \in [x_1,x_2]$, *where* $\zeta \in L[x_1,x_2]$. *Then for* $\sigma \in [x_1,x_2]$, *the inequality given below holds for Caputo derivatives:*

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^{C}D^{\alpha}_{\frac{x_1+x_2}{2}-}\Psi(x_1)+{}^{C}D^{\alpha}_{\frac{x_1+x_2}{2}+}\Psi(x_2)\right)-\Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)\right|$$
$$\leq \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^{C}D^{\alpha+1}_{\frac{x_1+x_2}{2}-}\zeta(x_1)+{}^{C}D^{\alpha+1}_{\frac{x_1+x_2}{2}+}\zeta(x_2)\right), \tag{18}$$

where $\alpha > 0$.

Proof: Under the given conditions, we have the following inequality

$$\int_{x_1}^{\frac{x_1+x_2}{2}}(\varphi-x_1)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi+\int_{x_1}^{\frac{x_1+x_2}{2}}(\varphi-x_1)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi \geq 0. \tag{19}$$

Integrating by parts, we have

$$\left(\frac{x_2-x_1}{2}\right)^{n-\alpha}\Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)-(n-\alpha)\frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha)}\int_{x_1}^{\frac{x_1+x_2}{2}}(\varphi-x_1)^{n-\alpha-1}\Psi^{(n)}(\varphi)\,\mathrm{d}\varphi$$
$$\geq -\int_{x_1}^{\frac{x_1+x_2}{2}}(\varphi-x_1)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi. \tag{20}$$

After some rearrangements of terms and using the definition of Caputo fractional derivatives, one can obtain the following fractional integral inequality

$$\frac{2^{n-\alpha}\Gamma(n-\alpha+1)^C D^{\alpha}_{\frac{x_1+x_2}{2}-}\Psi(x_1)}{(x_2-x_1)^{n-\alpha}} - \Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)$$
$$\leq \frac{2^{n-\alpha}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}} {}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}-}\zeta(x_1). \tag{21}$$

Under given conditions, one can also have the following inequality

$$\int_{\frac{x_1+x_2}{2}}^{x_2}(x_2-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi - \int_{\frac{x_1+x_2}{2}}^{x_2}(x_2-\varphi)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \geq 0. \tag{22}$$

From which we have

$$\left(\frac{x_2-x_1}{2}\right)^{n-\alpha}\Psi^{(n)}\left(\frac{x_1+x_2}{2}\right) - (n-\alpha)\frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha)}\int_{\frac{x_1+x_2}{2}}^{x_2}(x_2-\varphi)^{n-\alpha-1}\Psi^{(n)}(\varphi)\,\mathrm{d}\varphi$$
$$\geq -\int_{\frac{x_1+x_2}{2}}^{x_2}(x_2-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi. \tag{23}$$

After some rearrangements of terms and using the definition of Caputo fractional derivatives, one can obtain the following fractional integral inequality

$$\frac{2^{n-\alpha}\Gamma(n-\alpha+1)^C D^{\alpha}_{\frac{x_1+x_2}{2}+}\Psi(x_2)}{(x_2-x_1)^{n-\alpha}} - \Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)$$
$$\leq \frac{2^{n-\alpha}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}} {}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}+}\zeta(x_2). \tag{24}$$

By adding inequalities (21) and (24), the following inequality is obtained

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^C D^{\alpha}_{\frac{x_1+x_2}{2}-}\Psi(x_1) + {}^C D^{\alpha}_{\frac{x_1+x_2}{2}+}\Psi(x_2)\right) - \Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)$$
$$\leq \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}-}\zeta(x_1) + {}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}+}\zeta(x_2)\right). \tag{25}$$

Now, we interchange the non-negative terms $(\varphi - x_1)^{n-\alpha}$ and $(x_2-\varphi)^{n-\alpha}$ in inequalities (19) and (22). Then by adding the obtained inequalities after integration like inequalities (21) and (24), one can get the following fractional derivative inequality:

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^C D^{\alpha}_{\frac{x_1+x_2}{2}-}\Psi(x_1) + {}^C D^{\alpha}_{\frac{x_1+x_2}{2}+}\Psi(x_2)\right) - \Psi^{(n)}\left(\frac{x_1+x_2}{2}\right)$$
$$\geq -\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(x_2-x_1)^{n-\alpha}}\left({}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}-}\zeta(x_1) + {}^C D^{\alpha+1}_{\frac{x_1+x_2}{2}+}\zeta(x_2)\right). \tag{26}$$

The required inequality is the combination of inequalities (25) and (26). ∎

Theorem 2.3: *Let $\Psi : [x_1, x_2] \longrightarrow \mathbb{R}$ be a differentiable function which satisfies $\zeta(\varphi) \leq \Psi^{(n+1)}(\varphi) \leq \eta(\varphi)$ for all $\varphi \in [x_1, x_2]$, where $\zeta, \eta \in L[x_1, x_2]$. Then for $\sigma \in [x_1, x_2]$, we have the inequality given below:*

$$\begin{aligned}
&\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi \\
&\leq \Psi^{(n)}(\sigma) - \frac{\Gamma(n-\alpha+1)\left({}^{C}D^{\alpha}_{\sigma-}\Psi(x_1) + {}^{C}D^{\alpha}_{\sigma+}\Phi(x_2)\right)}{(\sigma - x_1)^{n-\alpha} + (x_2 - \sigma)^{n-\alpha}} \\
&\leq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi,
\end{aligned} \tag{27}$$

where $\alpha > 0$.

Proof: Let $\sigma \in [x_1, x_2]$. Then under the given conditions, the following inequalities hold:

$$\begin{aligned}
&\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \geq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi, \\
&\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \leq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi, \\
&\int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \geq \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi, \\
&\int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \leq \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi.
\end{aligned}$$

From above four inequalities, the following inequality holds:

$$\begin{aligned}
&\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi \\
&\leq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \\
&\leq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi.
\end{aligned} \tag{28}$$

This further gives the following inequality

$$\begin{aligned}
&\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi \\
&\leq \left((\sigma - x_1)^{n-\alpha} + (x_2 - \sigma)^{n-\alpha}\right) \Psi^{(n)}(\sigma) \\
&\quad - \Gamma(n-\alpha+1)\left({}^{C}D^{\alpha}_{\sigma-}\Psi(x_1) + {}^{C}D^{\alpha}_{\sigma+}\Phi(x_2)\right) \\
&\leq \int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi.
\end{aligned} \tag{29}$$

Hence the required inequality can be formulated. ■

Corollary 2.3: *By setting* $\sigma = \frac{x_1+x_2}{2}$ *in (27), the Caputo fractional Hadamard inequality is obtained as follows:*

$$\int_{x_1}^{\frac{x_1+x_2}{2}} (\varphi - x_1)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi - \int_{\frac{x_1+x_2}{2}}^{x_2} (x_2 - \varphi)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi$$
$$\leq \Psi^{(n)}\left(\frac{x_1+x_2}{2}\right) - \frac{2^{n-\alpha-1}\Gamma(\alpha+1)}{(x_2-x_1)^{n-\alpha}} \left({}^{C}D^{\alpha}_{\frac{x_1+x_2}{2}-}\Psi(x_1) + {}^{C}D^{\alpha}_{\frac{x_1+x_2}{2}+}\Phi(x_2)\right)$$
$$\leq \int_{x_1}^{\frac{x_1+x_2}{2}} (\varphi - x_1)^{n-\alpha} \eta(\varphi)\,\mathrm{d}\varphi - \int_{\frac{x_1+x_2}{2}}^{x_2} (x_2 - \varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi. \tag{30}$$

Next, we give the fractional version of Ostrowski-Grüss-type inequality using Caputo fractional derivatives:

Theorem 2.4: *Under the assumptions of Theorem 2.3, we have*

$$\left|\Psi^{(n)}(\sigma) - \frac{{}^{C}D^{\alpha}_{\sigma-}\Psi(u_1) + {}^{C}D^{\alpha}_{\sigma+}\Psi(u_2) + {}^{C}D^{\alpha}_{u_1+}\Psi(\sigma) + {}^{C}D^{\alpha}_{u_2-}\Psi(\sigma)}{(\Gamma(n-\alpha+1))^{-1}((\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha})}\right.$$
$$\left.+\frac{(\sigma-x_1)^{n-\alpha}\Psi^{(n)}(x_1) + (x_2-\sigma)^{n-\alpha}\Psi^{(n)}(x_2)}{(\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha}}\right|$$
$$\leq \frac{1}{(\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha}} \left[(\sigma-x_1)^{n-\alpha}\right.$$
$$\int_{x_1}^{\sigma} \eta(\varphi)\,\mathrm{d}\varphi + (x_2-\sigma)^{n-\alpha} \int_{\sigma}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi$$
$$\left.- \int_{\sigma}^{x_2} (x_2-\varphi)^{n-\alpha} \zeta(\varphi)\,\mathrm{d}\varphi\right]. \tag{31}$$

Proof: $\varphi \in [x_1, \sigma]$, using the non-negativity of $(\sigma-\varphi)^{n-\alpha}(\Psi^{(n+1)}(\varphi) - \zeta(\varphi))$ and $(\varphi - x_1)^{n-\alpha}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))$, we have

$$\int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha}(\Psi^{(n+1)}(\varphi) - \zeta(\varphi))\,\mathrm{d}\varphi + \int_{x_1}^{\sigma} (\varphi-x_1)^{n-\alpha}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))\,\mathrm{d}\varphi \geq 0.$$

This gives

$$\int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi + \int_{x_1}^{\sigma} (\varphi-x_1)^{n-\alpha}\eta(\varphi)\,\mathrm{d}\varphi$$
$$- \int_{x_1}^{\sigma} (\varphi-x_1)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \geq 0. \tag{32}$$

Integrating and using the inequality $(\varphi-x_1)^{n-\alpha} \leq (\sigma-x_1)^{n-\alpha}$ for $\varphi \in [x_1, \sigma]$, we have

$$(\sigma-x_1)^{n-\alpha}\left(\Psi^{(n)}(x_1) + \Psi^{(n)}(\sigma)\right) - \Gamma(n-\alpha+1)\left({}^{C}D^{\alpha}_{u_1+}\Psi(\sigma) + {}^{C}D^{\alpha}_{\sigma-}\Psi(u_1)\right)$$
$$\leq (\sigma-x_1)^{n-\alpha} \int_{x_1}^{\sigma} \eta(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi. \tag{33}$$

Similarly, for $\varphi \in [\sigma, x_2]$, we have

$$\int_{\sigma}^{x_2} (\varphi - \sigma)^{n-\alpha}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))\,\mathrm{d}\varphi + \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}(\Psi^{(n+1)}(\varphi) - \zeta(\varphi))\,\mathrm{d}\varphi \geq 0.$$

This gives

$$\begin{aligned} &\int_{\sigma}^{x_2} (\varphi - \sigma)^{n-\alpha}\eta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (\varphi - \sigma)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \\ &\quad + \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}\Psi^{(n+1)}(\varphi)\,\mathrm{d}\varphi \\ &- \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi \geq 0. \end{aligned} \tag{34}$$

Integrating and using the inequality $(\varphi - \sigma) \leq (x_2 - \sigma)$ for $\varphi \in [\sigma, x_2]$, we have

$$\begin{aligned} &(x_2 - \sigma)^{n-\alpha}(\Psi^{(n)}(x_2) + \Psi^{(n)}(\sigma)) - \Gamma(n - \alpha + 1)({}^{C}D^{\alpha}_{u_2-}\Psi(\sigma) + {}^{C}D^{\alpha}_{\sigma+}\Psi(u_2)) \\ &\leq (x_2 - \sigma)^{n-\alpha}\int_{\sigma}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi). \end{aligned} \tag{35}$$

By adding (34) and (36), we obtain:

$$\begin{aligned} &\Psi^{(n)}(\sigma) - \frac{{}^{C}D^{\alpha}_{\sigma-}\Psi(u_1) + {}^{C}D^{\alpha}_{\sigma+}\Psi(u_2) + {}^{C}D^{\alpha}_{u_1+}\Psi(\sigma) + {}^{C}D^{\alpha}_{u_2-}\Psi(\sigma)}{(\Gamma(n - \alpha + 1))^{-1}((\sigma - x_1)^{n-\alpha} + (x_2 - \sigma)^{n-\alpha})} \\ &+ \frac{(\sigma - x_1)^{n-\alpha}\Psi^{(n)}(x_1) + (x_2 - \sigma)^{n-\alpha}\Psi^{(n)}(x_2)}{(\sigma - x_1)^{n-\alpha} + (x_2 - \sigma)^{n-\alpha}} \\ &\leq \frac{1}{(\sigma - x_1)^{n-\alpha} + (x_2 - \sigma)^{n-\alpha}}\Bigg[(\sigma - x_1)^{n-\alpha} \\ &\int_{x_1}^{\sigma} \eta(\varphi)\,\mathrm{d}\varphi + (x_2 - \sigma)^{n-\alpha}\int_{\sigma}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{\sigma} (\sigma - \varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi \\ &\quad - \int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi\Bigg]. \end{aligned} \tag{36}$$

On the other hand, one can have:

$$\int_{x_1}^{\sigma} (\varphi - x_1)^{n-\alpha}(\Psi^{(n+1)}(\varphi) - \zeta(\varphi))\,\mathrm{d}\varphi + \int_{x_1}^{\sigma} (\sigma - \varphi)^{n-\alpha}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))\,\mathrm{d}\varphi \geq 0, \tag{37}$$

$$\int_{\sigma}^{x_2} (x_2 - \varphi)^{n-\alpha}(\eta(\varphi) - \Psi^{(n+1)}(\varphi))\,\mathrm{d}\varphi + \int_{\sigma}^{x_2} (\varphi - \sigma)^{n-\alpha}(\Psi^{(n+1)}(\varphi) - \zeta(\varphi))\,\mathrm{d}\varphi \geq 0. \tag{38}$$

Computing integrals and adding the resulting inequalities, we get

$$\Psi^{(n)}(\sigma) - \frac{{}^{C}D^{\alpha}_{\sigma-}\Psi(u_1) + {}^{C}D^{\alpha}_{\sigma+}\Psi(u_2) + {}^{C}D^{\alpha}_{u_1+}\Psi(\sigma) + {}^{C}D^{\alpha}_{u_2-}\Psi(\sigma)}{(\Gamma(n-\alpha+1))^{-1}((\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha})}$$

$$+ \frac{(\sigma-x_1)^{n-\alpha}\Psi^{(n)}(x_1) + (x_2-\sigma)^{n-\alpha}\Psi^{(n)}(x_2)}{(\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha}}$$

$$\geq \frac{-1}{(\sigma-x_1)^{n-\alpha} + (x_2-\sigma)^{n-\alpha}} \Bigg[(\sigma-x_1)^{n-\alpha}$$

$$\int_{x_1}^{\sigma} \eta(\varphi)\,\mathrm{d}\varphi + (x_2-\sigma)^{n-\alpha}\int_{\sigma}^{x_2} \eta(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{\sigma} (\sigma-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi$$

$$- \int_{\sigma}^{x_2} (x_2-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi\Bigg]. \tag{39}$$

The required inequality (40) can be obtained from (36) and (39). ∎

Corollary 2.4: *By setting $\sigma = x_1$ in* (40), *we get Caputo fractional Hadamard-type inequality as follows:*

$$\left|\frac{\Psi^{(n)}(x_1) + \Psi^{(n)}(x_2)}{2} - \frac{\Gamma(n-\alpha+1)({}^{C}D^{\alpha}_{u_2-}\Psi(u_1) + {}^{C}D^{\alpha}_{u_1+}\Psi(u_2))}{2(x_2-x_1)^{n-\alpha}}\right|$$

$$\leq \frac{1}{2(x_2-x_1)^{n-\alpha}}\left[(x_2-x_1)^{n-\alpha}\int_{x_1}^{x_2}\eta(\varphi)\,\mathrm{d}\varphi - \int_{x_1}^{x_2}(x_2-\varphi)^{n-\alpha}\zeta(\varphi)\,\mathrm{d}\varphi\right]. \tag{40}$$

3. Conclusion

We derived Ostrowski- and Hadamard-type inequalities for Caputo fractional derivatives. Conditions of well-known inequalities were utilized in unusual ways . In the recent past, such inequalities were established by establishing identities with the help of convenient kernels. Also, the methods of proofs can be extended to generalized fractional integrals and derivatives.

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References

[1] Ostrowski A. Über die Absolutabweichung einer dierentierbaren Funktion von ihren Integralmittelwert. Comment Math Helv. 1938;10:226–227. doi: 10.1007/BF01214290
[2] Grüss G. Über das maximum des absolten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{b-a}^2 \int_a^b f(x)\,\mathrm{d}x \int_a^b f(x)\,\mathrm{d}x$. Math Z. 1935;39:215–226. doi: 10.1007/BF01201355
[3] Dragomir SS, Wang S. An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and some numerical quadrature rules. Comput Math Appl. 1997;33:15–20. doi: 10.1016/S0898-1221(97)00084-9
[4] Dragomir SS, Wang S. Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules. Appl Math Lett. 1998;11(1):105–109. doi: 10.1016/S0893-9659(97)00142-0
[5] Dragomir SS, Rassias TM, editors. Ostrowski-type inequalities and applications in numerical integration. Dordrecht, Boston, London: Kluwer academic publishers; 2002.
[6] Milovanović GV, Pečarić JE. On generalization of the inequality of A. Ostrowski and some related applications. Univ Beograd Publ Elektrotehn Fak Ser Mat Fiz. 1976;544–576:155–158.
[7] Ujević N. Some double integral inequalities and applications. Acta Math Univ Comen. 2002;2:189–199.
[8] Kermausuor S. Ostrowski type inequalities for functions whose derivatives are strongly (α, m)-convex via k-Riemann-Liouville fractional integrals. Stud Univ Babes-Bolyai Math. 2019;64:25–34. doi: 10.24193/subbmath
[9] Gürbüz M, Taşdan Y, Set E. Ostrowski type inequalities via the Katugampola fractional integrals. AIMS Math. 2020;5:42–53. doi: 10.3934/math.2020004
[10] Set E, Akdemir AO, Gözpınar A, et al. Ostrowski type inequalities via new fractional conformable integrals. AIMS Math. 2019;4:1684–1697. doi: 10.3934/math.2019.6.1684
[11] Mubeen S, Iqbal S. Grüss type integral inequalities for generalized Riemann-Liouville k-fractional integrals. J Inequal Appl. 2016;2016:109. doi: 10.1186/s13660-016-1052-x
[12] Rahman G, Nisar KS, Qi F. Some new inequalities of the Grüss type for conformable fractional integrals. AIMS Math. 2018;3:575–583. doi: 10.3934/Math.2018.4.575
[13] Farid G, Rehman AU, Mishra VN, et al. Fractional integral inequalities of Grüss type via generalized Mittag-Leffler function. Int J Anal Appl. 2019;17:548–558.
[14] Farid G. Straightforward proofs of Ostrowski inequality and some related results. Int J Anal. 2016;2016:Article ID 3918483, 5 pp.
[15] Dragomir SS, Pearce CEM. Selected topics on Hermite-Hadamard inequalities and applications. RGMIA Monographs, Victoria University; 2000. Available from: https://ajmaa.org/RGMIA/monographs.php/.
[16] Sarikaya MZ, Set E, Yaldiz H, et al. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math Comput Model. 2013;57(9–10):2403–2407. doi: 10.1016/j.mcm.2011.12.048
[17] Sarikaya MZ, Yildirim H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. Miskolc Math Notes. 2017;17(2):1049–1059. doi: 10.18514/MMN.2017.1197
[18] Kunt M, İşcan İ. Hermite-Hadamard type inequalities for p-convex functions via fractional integrals. Moroc J Pure Appl Anal. 2017;3(1):22–35. doi: 10.1515/mjpaa-2017-0003
[19] Khan TU, Khan MA. Hermite-Hadamard inequality for new generalized conformable fractional operators. AIMS Math. 2020;6:23–38. doi: 10.3934/math.2021002
[20] Mohammed PO, Sarikaya MZ, Baleanu D. On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals. Symmetry. 2020;12(4):595. doi: 10.3390/sym12040595
[21] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier; 2006.
[22] Columbu A, Díaz Fuentes R, Frassu S. Uniform-in-time boundedness in a class of local and nonlocal nonlinear attraction–repulsion chemotaxis models with logistics. Nonlinear Anal: Real World Appl. 2024;79:104135. doi: 10.1016/j.nonrwa.2024.104135

[23] Li T, Frassu S, Viglialoro G. Combining effects ensuring boundedness in an attraction–repulsion chemotaxis model with production and consumption. Z fur Angew Math Phys. 2023;74(3):109. doi: 10.1007/s00033-023-01976-0

[24] Frassu S, Viglialoro G. Boundedness for a fully parabolic Keller-Segel model with sublinear segregation and superlinear aggregation. Acta Appl Math. 2021;171(1):1–20. doi: 10.1007/s10440-021-00386-6

[25] Ishida S, Lankeit J, Viglialoro G. A Keller–Segel type taxis model with ecological interpretation and boundedness due to gradient nonlinearities. Discrete Contin Dyn Syst-B. 2024;29(9):3955–3969.