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Inequalities for Products of Two Kinds of Convexities and Consequent Results

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ABSTRACT

The product of two convex functions is convex under certain conditions, which have motivated extensions to generalized convexities. In the present paper, we establish new Hermite–Hadamard–type inequalities for the product of m-convex and (α, m)-convex functions. Through the presentation of results for these extended classes, we arrive at inequalities that include classical cases as special cases (e.g., when $m = 1$ or $\alpha = 1$). Our results improve and extend already established inequalities to find wider applications in convex analysis and optimization.

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1 | Introduction

In recent years, remarkable attention has been given to studying and investigating various aspects of classical concept of the convexity. Resultantly, this concept has been extended and generalized in many directions. A function earns the label “convex” when its graph consistently lies above the straight line connecting any two of its points, a concept that has been widely explored and refined over the years [1]. The classical or the usual convexity is defined as follows:

Definition 1. A function $\zeta\colon I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on interval I, if

$$\zeta(t\mu + (1-t)\nu) \leq t\zeta(\mu) + (1-t)\zeta(\nu), \tag{1}$$

for all $\mu, \nu \in I$ and $t \in [0, 1]$.

Convexity in connection with integral inequalities is an interesting field of research, since many inequalities are direct consequences of the applications of convex functions. Convexity of functions is a fundamental topic in the subject of mathematical analysis, finding extensive applications across various domains, including economics, engineering, and optimization and many other fields [2, 3]. In [4], Toseef et al. introduced a family of quadrature formulas and analyzed their associated error bounds for convex functions. Furthermore, Murota and Tamura do the comprehensive survey of recent results on integrally convex functions with some new technical results [5]. Recently, Correa et al. introduce a fresh perspective on convex analysis and optimization, rooted in a thorough exploration of how the supremum of families of convex functions behaves and what properties it exhibits [6]. Defining the Jensen–Mercer inequality in terms of coordinates posed a significant challenge for researchers in the field of inequalities. However, Toseef and Aamir have successfully established the Jensen–Mercer inequality specifically for coordinated convex functions in [7]. Min–max problems have broad applications in machine learning. Hassan Rafique et al. introduced a family of nonconvex min–max problems, whose objective function is weakly convex in the variables of minimization and is concave in the variables of maximization in [8].

In recent times, there has been a growing interest in exploring new types of convex functions among researchers. The authors delved into the properties of convex functions, which represent a generalization of classical convexity [9]. This concept extended into s-convex of second type and yielded a host of novel Hermite–Hadamard–type inequalities [10, 11]. In another direction, the authors delved into the concept of m-convex functions and the consequent properties compatible with convexity [12]. After some modification, the m-convex function was developed into (α, m)-convex function and proved Hermite–Hadamard–type inequalities [13]. The exploration of convex functions originated from Jensen's pioneering ideas, and significant contributions have been made by many authors [14, 15].

The Hermite–Hadamard inequality is the most significant result of the convex analysis which provides necessary and sufficient conditions for a function to be convex [16]. Its extensions have been studied extensively, including the Hermite–Hadamard–type inequalities for different classes of functions such as m-convex, h-convex, and (α, m)-convex functions [17, 18]. The well famous result of Hermite–Hadamard inequality is given as follows:

Let $\zeta: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $\mu, \nu \in I$ with $\mu < \nu$. The following inequality holds:

$$\zeta\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)dx \leq \frac{\zeta(\mu)+\zeta(\nu)}{2}. \tag{2}$$

In a separate context, the authors established Hermite–Hadamard–type inequalities pertinent to the product of two convex functions defined over coordinates within a rectangular plane, providing a foundation for subsequent explorations [19]. Similarly, this product extended these inequalities to various types of convex functions, thereby widening their scope [20–25]. From some latest articles, it is found that superquadraticity is strong refinement of convexity. Recently, the authors have significantly advanced the theory of mathematical inequalities by introducing several further refinements of superquadraticity and convexity; we refer to reader [26–29] and references therein.

We will establish some Hermite–Hadamard–type inequalities for the product of m-convex and (α, m)-convex functions.

Let us give the definitions of m-convex and (α, m)-convex functions along with some related results.

Definition 2. A function $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ is said to be m-convex if the inequality

$$\zeta(t\mu + t_m\nu) \leq t\zeta(\mu) + t_m\zeta(\nu), \tag{3}$$

where $t_m = m(1-t)$ holds for all $\mu, \nu \in [0, \infty)$, $t \in [0, 1]$, and $m \in (0, 1]$.

In future proofs, we will utilize t_m to establish the theorems.

Definition 3. A function $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ is said to be (α, m)-convex if the inequality

$$\zeta(t\mu + t_m\nu) \leq \left(t^\alpha\zeta(\mu) + t_m^\alpha\zeta(\nu)\right), \tag{4}$$

where $t_m = m(1-t)$, $t_m^\alpha = m(1-t^\alpha)$ holds for all $\mu, \nu \in [0, \infty)$, $\alpha \in [0, 1]$, and $m \in (0, 1]$.

Dragomir and Toader proved following inequalities of Hermite–Hadamard inequalities type for m-convex functions in [30].

Theorem 1. *Let $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ be a m-convex function with $m \in (0, 1]$. If $0 \leq \mu < \nu < \infty$, and $\zeta \in L_1[\mu, \nu]$, then one has the inequality:*

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)dx \leq \min\left\{\frac{\zeta(\mu)+m\zeta(\nu/m)}{2}, \frac{\zeta(\nu)+m\zeta(\mu/m)}{2}\right\}. \tag{5}$$

Theorem 2. *Let $\zeta: [0, 1) \longrightarrow \mathbb{R}$ be a m-convex function with $m \in (0, 1]$. If $0 \leq \mu < \nu < \infty$ and ζ is differentiable on $(0, \infty)$, then one has the inequality:*

$$\begin{aligned}\frac{f(m\nu)}{m} - \frac{\nu-\mu}{2}\zeta'(m\nu) &\leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)dx \\ &\leq \frac{(\nu-m\mu)\zeta(\nu)-(\mu-m\nu)\zeta(\mu)}{2(\nu-\mu)}.\end{aligned} \tag{6}$$

Dragomir and Pearce proved the following Hermite–Hadamard–type inequalities for m-convex functions in [31].

Theorem 3. *Let $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ be a m-convex function with $m \in (0, 1]$. If $0 \leq \mu < \nu$, and $\zeta \in L_1[\mu, \nu]$, then one has the inequality:*

$$\zeta\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\frac{\zeta(x)+m\zeta(x/m)}{2}dx. \tag{7}$$

In [32], Set et al. proved the following Hermite–Hadamard–type inequalities for (a, m)-convex functions.

Theorem 4. *Let $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ be a (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq \mu < \nu < \infty$ and $\zeta \in L_1[\mu, \nu] \cap L_1[\mu/m, \nu/m]$, then the following inequality holds:*

$$\zeta\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\frac{\zeta(x)+m(2^\alpha-1)\zeta(x/m)}{2^\alpha}dx. \tag{8}$$

Theorem 5. *Let $\zeta: [0, \infty) \longrightarrow \mathbb{R}$ be a (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq \mu < \nu < \infty$, and $\zeta \in L_1[\mu, \nu]$, then one has the inequalities:*

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)dx \leq \min\left\{\frac{\zeta(\mu)+m\alpha\zeta(\nu/m)}{\alpha+1}, \frac{\zeta(\nu)+m\alpha\zeta(\mu/m)}{\alpha+1}\right\}, \tag{9}$$

and

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)dx \leq \frac{1}{2}\left[\frac{\zeta(\mu)+\zeta(\nu)+m\alpha\zeta(\mu/m)+m\alpha\zeta(\nu/m)}{\alpha+1}\right]. \tag{10}$$

In the following, we give results which are motivations of this work. These are proved for convex functions. We state them in the following theorems. Pachpatte proved the following Hermite–Hadamard–type inequalities for the product of convex functions in [33].

Theorem 6. *Let ζ and ξ be real-valued, non-negative, and convex functions on $[\mu, \nu]$, then one has the inequalities:*

$$\frac{1}{\nu-\mu}\int_{\mu}^{\nu}\zeta(x)\xi(x)dx \leq \frac{1}{3}M(\mu, \nu) + \frac{1}{6}N(\mu, \nu), \tag{11}$$

and

$$2\zeta\Big(\frac{\mu+\nu}{2}\Big)\xi\Big(\frac{\mu+\nu}{2}\Big) \le \frac{1}{\nu-\mu}\int_\mu^\nu \zeta(x)\xi(x)dx + \frac{1}{6}M(\mu,\nu) + \frac{1}{3}N(\mu,\nu), \tag{12}$$

where
$M(\mu,\nu) = \zeta(\mu)\xi(\mu) + \zeta(\nu)\xi(\nu), N(\mu,\nu) = \zeta(\mu)\xi(\nu) + \xi(\mu)\zeta(\nu)$.

Theorem 7. *Let ζ and ξ be real-valued, non-negative, and convex functions on $[\mu,\nu]$, then one has the inequalities:*

$$\begin{aligned}&\frac{3}{2(\nu-\mu)^2}\int_\mu^\nu\int_\mu^\nu\int_0^1 \zeta(t\mu+(1-t)\nu).\xi(t\mu+(1-t)\nu)dtd\nu d\mu\\ &\le \frac{1}{\nu-\mu}\int_\mu^\nu \zeta(x)\xi(x)dx + \frac{1}{8}\Big[\frac{M(\mu,\nu)+N(\mu,\nu)}{(\nu-\mu)^2}\Big],\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\frac{3}{\nu-\mu}\int_\mu^\nu\int_0^1 \zeta\Big(tx+(1-t)\Big(\frac{\mu+\nu}{2}\Big)\Big)\xi\Big(tx+(1-t)\Big(\frac{\mu+\nu}{2}\Big)\Big)dtdx\\ &\le \frac{1}{\nu-\mu}\int_\mu^\nu \zeta(x)\xi(x)dx + \frac{1}{4}\Big(\frac{1+\nu-\mu}{\nu-\mu}\Big)[M(\mu,\nu)+N(\mu,\nu)].\end{aligned} \tag{14}$$

In the upcoming sections, we have investigated and proved analog results to Theorem 6 and Theorem 7 by applying m- and (α,m)-convexities. Results of these sections explore new inequalities for the product of m-convex functions as well as the product of (α,m)-convex functions. Classical results are particular cases of our findings in this paper.

2 | Main Results

The main results are composed in the form of following theorems.

Theorem 8. *Let $\zeta,\xi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a m-convex function. If $0 \le \mu < \nu < \infty$, and $\zeta,\xi \in L_1[\mu,\nu]$, then one has the inequalities:*

$$\frac{1}{m\nu-\mu}\int_\mu^{m\nu} \zeta(x)\xi(x)dx \le \frac{1}{3}\zeta(\mu)\xi(\mu) + \frac{m^2}{3}\zeta(\nu)\xi(\nu) + \frac{m}{6}N(\mu,\nu), \tag{15}$$

and

$$\begin{aligned}&2\zeta\Big(\frac{\mu+m\nu}{2}\Big)\xi\Big(\frac{\mu+m\nu}{2}\Big)\\ &\le \frac{1}{m\nu-\mu}\int_\mu^{m\nu} \zeta(x)\xi(x)dx + \frac{m}{6}M(\mu,\nu) + \Big(\frac{m^2+1}{6}\Big)N(\mu,\nu),\end{aligned} \tag{16}$$

where
$M(\mu,\nu) = \zeta(\mu)\xi(\mu) + \zeta(\nu)\xi(\nu), N(\mu,\nu) = \zeta(\mu)\xi(\nu) + \xi(\mu)\zeta(\nu)$.

Proof 1. Since ζ and ξ are m-convex on $[\mu,\nu]$, then for t in [0,1], we have

$$\begin{aligned}\zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu) &\le t^2\zeta(\mu)\xi(\mu) + mt(1-t)\zeta(\mu)\xi(\nu),\\ &+ mt(1-t)\xi(\mu)\zeta(\nu) + m^2(1-t)^2\zeta(\nu)\xi(\nu).\end{aligned} \tag{17}$$

The above inequality in the integral form is obtained as follows:

$$\begin{aligned}&\int_0^1 \zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu)dt \le \zeta(\mu)\xi(\mu)\int_0^1 t^2dt + m^2\zeta(\nu)\xi(\nu)\int_0^1(1-t)^2dt\\ &+ m(\zeta(\mu)\xi(\nu)+\xi(\mu)\zeta(\nu))\int_0^1 t(1-t)dt = \frac{1}{3}\zeta(\mu)\xi(\mu) + \frac{m^2}{3}\zeta(\nu)\xi(\nu) + \frac{m}{6}N(\mu,\nu).\end{aligned} \tag{18}$$

By substituting $x = t\mu + t_m\nu$ on the left side of the above inequality, it is easy to observe

$$\frac{1}{m\nu-\mu}\int_\mu^{m\nu} \zeta(x)\xi(x)dx \le \frac{1}{3}\zeta(\mu)\xi(\mu) + \frac{m^2}{3}\zeta(\nu)\xi(\nu) + \frac{m}{6}N(\mu,\nu). \tag{19}$$

The proof of (15) is completed. Since ζ and ξ are m-convex on $[\mu,\nu]$, then for $t \in [0,1]$, we observe that

$$\begin{aligned}&\zeta\Big(\frac{\mu+m\nu}{2}\Big)\xi\Big(\frac{\mu+m\nu}{2}\Big)\\ &\le \frac{1}{4}[(\zeta(t\mu+t_m\nu)+\zeta((1-t)\mu+mt\nu))(\xi(t\mu+t_m\nu)+\xi((1-t)\mu+mt\nu))]\\ &\le \frac{1}{4}[(t\mu+t_m\nu)(\xi(t\mu+t_m\nu)) + (\zeta((1-t)\mu+mt\nu)\xi((1-t)\mu+mt\nu))]\\ &+ \frac{1}{4}[(t\zeta(\mu)+t_m\zeta(\nu))(t_m\xi(\mu)+t\xi(\nu)) + (t_m\zeta(\mu)+t\zeta(\nu))(t\xi(\mu)+t_m\xi(\nu))]\\ &= \frac{1}{4}[(t\mu+t_m\nu)(\xi(t\mu+t_m\nu)) + (\zeta(t_m\mu+t\nu)\xi(t_m\mu+t\nu))]\\ &+ \frac{1}{4}\big[2t_mt\zeta(\mu)\xi(\mu) + \big(t^2+m^2(1-t)^2\big)\zeta(\mu)\xi(\nu) + \big(t^2+m^2(1-t)^2\big)\zeta(\nu)\xi(\mu)\\ &+ 2t_mt\zeta(\nu)\xi(\nu)\big].\end{aligned} \tag{20}$$

After some simplification, the above inequality in the integral form is obtained as follows:

$$
\begin{aligned}
&\int_0^1 \zeta\Big(\frac{\mu+m\nu}{2}\Big)\xi\Big(\frac{\mu+m\nu}{2}\Big)dt \\
&\le \frac{1}{4}\Bigg[\int_0^1 \{(\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+(\zeta(1-t)\mu+mt\nu)\xi((1-t)\mu+mt\nu)\}dt\Bigg] \\
&\quad+\frac{m(\zeta(\mu)\xi(\mu)+\zeta(\nu)\xi(\nu))}{2}\int_0^1 t(1-t)dt+\frac{\zeta(\mu)\xi(\nu)+\zeta(\nu)\xi(\mu)}{4}\int_0^1 (t^2+m^2(1-t)^2)dt \\
&=\frac{1}{4}\Bigg[\int_0^1 \{(\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+(\zeta(t_m\mu+t\nu)\xi(t_m\mu+t\nu))\}dt+\frac{m}{12}M(\mu,\nu)\Bigg] \\
&\quad+\frac{1}{16}\Big(\frac{m^2+1}{3}\Big)N(\mu,\nu) \\
&=\frac{1}{2}\int_0^1 \zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu)dt+\frac{m}{12}M(\mu,\nu)+\Big(\frac{m^2+1}{12}\Big)N(\mu,\nu).
\end{aligned}
\tag{21}
$$

Now, multiplying by 2 and using the upcoming identity

$$\int_0^1 \zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu)dt=\frac{1}{m\nu-\mu}\int_\mu^{m\nu}\zeta(x)\xi(x)dx, \tag{22}$$

one can get the second inequality as required. The proof of (16) is completed. □

Remark 1. By putting $m=1$, we obtain the first and second inequality of Theorem 6.

Theorem 9. *Let $\zeta,\xi\colon [0,\infty)\longrightarrow\mathbb{R}$ be a m-convex function . If $0\le\mu<\nu<\infty$ and $\zeta,\xi\in L_1[\mu,\nu]$, then one has the inequalities:*

$$
\begin{aligned}
&\frac{3}{2(\nu-\mu)^2}\int_\mu^\nu\int_\mu^\nu\int_0^1 \zeta(tx+t_my).\xi(tx+t_my)dtdxdy \\
&\le \frac{m^2+1}{2(\nu-\mu)}\int_\mu^\nu \zeta(x)\xi(x)dx+\frac{m}{8}\Bigg[\frac{M(\mu,\nu)+N(\mu,\nu)}{(\nu-\mu)^2}\Bigg],
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\frac{3}{\nu-\mu}\int_\mu^\nu\int_0^1 \zeta\Big(tx+t_m\Big(\frac{\mu+\nu}{2}\Big)\Big)\xi\Big(tx+t_m\Big(\frac{\mu+\nu}{2}\Big)\Big)dtdx \\
&\le \frac{1}{\nu-\mu}\int_\mu^\nu \zeta(x)\xi(x)dx+\frac{1}{4}\Big(\frac{m^2(\nu-\mu)+m}{(\nu-\mu)}\Big)(M(\mu,\nu)+N(\mu,\nu)),
\end{aligned}
\tag{24}
$$

where $M(\mu,\nu)$ and $N(\mu,\nu)$ are defined in Theorem 8.

Proof 2. Since ζ and ξ are m-convex on $[\mu,\nu]$, then for $x,y\in[\mu,\nu]$ and t $\in[0,1]$, we have

$$\zeta(tx+t_my).\xi(tx+t_my)\le t^2\zeta(x)\xi(x)+m^2(1-t)^2\zeta(y)\xi(y)+t_mt(\zeta(x)\xi(y)+\zeta(y)\xi(x)). \tag{25}$$

As explained in the proof of inequality given above, integrate over [0,1]

$$
\begin{aligned}
\int_0^1 \zeta(tx+t_my).\xi(tx+t_my)dt &\le \zeta(x)\xi(x)\int_0^1 t^2dt+m^2\zeta(y)\xi(y)\int_0^1(1-t)^2dt \\
&\quad+m(\zeta(x)\xi(y)+\xi(x)\zeta(y))\int_0^1 t(1-t)dt \\
&=\frac{1}{3}\zeta(x)\xi(x)+\frac{m^2}{3}\zeta(y)\xi(y)+\frac{m}{6}(\zeta(x)\xi(y)+\xi(x)\zeta(y)).
\end{aligned}
\tag{26}
$$

Now, integrating both sides on $[\mu,\nu]\times[\mu,\nu]$, we obtain

$$\int_\mu^\nu\int_\mu^\nu\int_0^1 \zeta(tx+t_m y).\xi(tx+t_m y)dxdydt \leq \frac{(m^2+1)(\nu-\mu)}{3}\int_\mu^\nu \zeta(x)\xi(x)dx + \frac{m}{12}(M(\mu,\nu)+N(\mu,\nu)). \tag{27}$$

By multiplying both sides with $3/(2(\nu-\mu)^2)$, one can obtain the required inequality (2.4). Since ζ and ξ are m-convex on $[\mu,\nu]$, we have the following inequalities:

$$\zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right) \leq t\zeta(x)+t_m\zeta\left(\frac{\mu+\nu}{2}\right) \tag{28}$$

$$\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right) \leq t\xi(x)+t_m\xi\left(\frac{\mu+\nu}{2}\right). \tag{29}$$

From (28) and (29), we have the following inequality

$$\begin{aligned}\zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right) &\leq t^2\zeta(x)\xi(x)+m^2(1-t)^2\zeta\left(\frac{\mu+\nu}{2}\right)\\ &\quad \xi\left(\frac{\mu+\nu}{2}\right)+t_m t\left[\zeta(x)\xi\left(\frac{\mu+\nu}{2}\right)+\zeta\left(\frac{\mu+\nu}{2}\right)\xi(x)\right].\end{aligned} \tag{30}$$

The above inequality in the integral form is as follows:

$$\begin{aligned}\int_0^1 \zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right) &\leq \zeta(x)\xi(x)\int_0^1 t^2dt + m^2\zeta\left(\frac{\mu+\nu}{2}\right)\\ &\quad\times\xi\left(\frac{\mu+\nu}{2}\right)\int_0^1(1-t)^2dt + m\left[\zeta(x)\xi\left(\frac{\mu+\nu}{2}\right)+\zeta\left(\frac{\mu+\nu}{2}\right)\xi(x)\right]\int_0^1 t(1-t)dt\\ &= \frac{1}{3}\zeta(x)\xi(x)+\frac{m^2}{3}\zeta\left(\frac{\mu+\nu}{2}\right)\xi\left(\frac{\mu+\nu}{2}\right)+\frac{m}{6}\left(\zeta(x)\xi\left(\frac{\mu+\nu}{2}\right)+\xi(x)\left(\frac{\mu+\nu}{2}\right)\right).\end{aligned} \tag{31}$$

As explained in the proof of previously solved inequality given above, the product of ζ and ξ is integrable on $[\mu,\nu]$. Now, integrating both sides of the above inequality on $[\mu,\nu]$, using the right half of the Hermite–Hadamard–type inequalities given above and convexity of ζ and ξ, we observe that

$$\begin{aligned}&\int_\mu^\nu\int_0^1 \zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)dtdx\\ &\leq \frac{1}{3}\int_\mu^\nu \zeta(x)\xi(x)dx+\frac{m^2}{3}(\nu-\mu)\zeta\left(\frac{\mu+\nu}{2}\right)\xi\left(\frac{\mu+\nu}{2}\right)\\ &\quad+\frac{m}{6}\left(\xi\left(\frac{\mu+\nu}{2}\right)\int_\mu^\nu\zeta(x)dx+\zeta\left(\frac{\mu+\nu}{2}\right)\int_\mu^\nu\xi(x)dx\right)\\ &\leq \frac{1}{3}\int_\mu^\nu \zeta(x)\xi(x)dx+\frac{m^2}{3}(\nu-\mu)\left(\frac{\zeta(\mu)+\zeta(\nu)}{2}\right)\left(\frac{\xi(\mu)+\xi(\nu)}{2}\right)\\ &\quad+\frac{m}{6}\left[\left(\frac{\zeta(\mu)+\zeta(\nu)}{2}\right)\left(\frac{\xi(\mu)+\xi(\nu)}{2}\right)+\left(\frac{\zeta(\mu)+\zeta(\nu)}{2}\right)\left(\frac{\xi(\mu)+\xi(\nu)}{2}\right)\right]\\ &= \frac{1}{3}\int_\mu^\nu \zeta(x)\xi(x)dx+\frac{m^2}{12}(\nu-\mu)(M(\mu,\nu)+N(\mu,\nu))+\frac{m}{12}(M(\mu,\nu)+N(\mu,\nu)).\end{aligned} \tag{32}$$

By multiplying both sides with $3/(\nu-\mu)$, one can obtain the required inequality (2.5). □

Remark 2. By putting $m=1$, we obtain the first and second inequalities of Theorem 7.

Theorem 10. *Let $\zeta, \xi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a (α, m)-convex function. If $0 \le \mu < \nu < \infty$, and $\zeta, \xi \in L_1[\mu,\nu]$, then one has the inequalities:*

$$\frac{1}{m\nu-\mu}\int_{\mu}^{m\nu}\zeta(x)\xi(x)dx \le \frac{\zeta(\mu)\xi(\mu)}{2\alpha+1} + \frac{2\alpha^2 m^2\zeta(\nu)\xi(\nu)}{(2\alpha+1)(\alpha+1)} + \frac{m\alpha N(\mu,\nu)}{(2\alpha+1)(\alpha+1)}, \tag{33}$$

and

$$\begin{aligned} &2\zeta\left(\frac{\mu+m\nu}{2}\right)\xi\left(\frac{\mu+m\nu}{2}\right) \\ &\le \frac{1}{m\nu-\mu}\int_{\mu}^{m\nu}\zeta(x)\xi(x)dx + \frac{2\alpha\big(\zeta(\mu)\xi(\mu)+m^2\zeta(\nu)\xi(\nu)\big)+m\big(2\alpha^2+\alpha+1\big)N(\mu,\nu)}{2(\alpha+1)(2\alpha+1)}, \end{aligned} \tag{34}$$

where $M(\mu,\nu)$ and $N(\mu,\nu)$ are same as defined in Theorem 8.

Proof 3. Since ζ and ξ are (α, m) convex on $[\mu,\nu]$, for $t \in [0,1]$, one can have

$$\begin{aligned} \zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu) &\le t^{2\alpha}\zeta(\mu)\xi(\mu) \\ &+ mt^{\alpha}(1-t^{\alpha})\zeta(\mu)\xi(\nu) + mt^{\alpha}(1-t^{\alpha})\xi(\mu)\xi(\nu) + m^2(1-t^{\alpha})^2\zeta(\nu)\xi(\nu) \\ &= t^{2\alpha}\zeta(\mu)\xi(\mu) + m(\zeta(\mu)\xi(\nu)+\xi(\mu)\zeta(\nu))t^{\alpha}(1-t^{\alpha}) + m^2(1-t^{\alpha})^2\zeta(\nu)\xi(\nu). \end{aligned} \tag{35}$$

The above inequality in the integral form is obtained as follows:

$$\begin{aligned} \int_0^1 \zeta(t\mu+t_m\nu)g(t\mu+t_m\nu)dt &\le \zeta(\mu)\xi(\mu)\int_0^1 t^{2\alpha}dt + m^2\zeta(\nu)\xi(\nu)\int_0^1(1-t^{\alpha})^2dt \\ &+ m(\zeta(\mu)\xi(\nu)+\xi(\mu)\zeta(\nu))\int_0^1 t^{\alpha}(1-t^{\alpha})dt \\ &= \frac{\zeta(\mu)\xi(\mu)}{2\alpha+1} + \frac{2\alpha^2m^2\zeta(\nu)\xi(\nu)}{(2\alpha+1)(\alpha+1)} + \frac{m\alpha(\zeta(\mu)\xi(\nu)+\xi(\mu)\zeta(\nu))}{(2\alpha+1)(\alpha+1)}, \end{aligned} \tag{36}$$

which is further simplified as follows:

$$\int_0^1 \zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu)dt \le \frac{\zeta(\mu)\xi(\mu)}{2\alpha+1} + \frac{2\alpha^2m^2\zeta(\nu)\xi(\nu)}{(2\alpha+1)(\alpha+1)} + \frac{m\alpha N(\mu,\nu)}{(2\alpha+1)(\alpha+1)}. \tag{37}$$

By using Identity (22) in (37), we obtain the required inequality (33). Since ζ and ξ are (α, m)-convex on $[\mu,\nu]$, for t $\in[\mu,\nu]$, we observe that

$$\begin{aligned} &\zeta\left(\frac{\mu+m\nu}{2}\right)\xi\left(\frac{\mu+m\nu}{2}\right) \\ &= \zeta\left(\frac{t\mu+t_m\nu}{2}+\frac{(1-t)\mu+mt\nu}{2}\right)\xi\left(\frac{t\mu+t_m\nu}{2}+\frac{(1-t)\mu+mt\nu}{2}\right) \\ &\le \frac{1}{4}[(\zeta(t\mu+t_m\nu)+\zeta((1-t)\mu+mt\nu))(\xi(t\mu+t_m\nu)+\xi((1-t)\mu+mt\nu))] \end{aligned}$$

$$\le \frac{1}{4}[(\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+\zeta((1-t)\mu+mt\nu)\xi((1-t)\mu+mt\nu))]$$
$$+\frac{1}{4}[(t^{\alpha}\zeta(\mu)+m(1-t^{\alpha})\zeta(\nu))((1-t^{\alpha})\xi(\mu)+mt^{\alpha}\xi(\nu))]$$
$$+\frac{1}{4}[(1-t^{\alpha})\zeta(\mu)+mt^{\alpha}\zeta(nu))(t^{\alpha}\xi(\mu)+m(1-t^{\alpha})\xi(\nu))]$$
$$=\frac{1}{4}[\{\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+(\zeta(1-t)\mu+mt\nu)\xi(1-t)\mu+mt\nu)]$$
$$+\frac{1}{4}\left(2t^{\alpha}(1-t^{\alpha})\zeta(\mu)\xi(\mu)+m\left(t^{2\alpha}+(1-t^{\alpha})^2\right\}\right)\zeta(\mu)\xi(\nu)$$
$$+m\left(t^{2\alpha}+(1-t^{\alpha})^2\right)\xi(\mu)\zeta(\nu)+(2m^2t^{\alpha}(1-t^{\alpha})\}\zeta(\nu)\xi(\nu)]. \tag{38}$$

After some simplification, integrating over [0,1], one can obtain the upcoming inequality

$$\begin{aligned}
&\int_0^1\zeta\left(\frac{\mu+m\nu}{2}\right)\xi\left(\frac{\mu+m\nu}{2}\right)dt\\
&\le \frac{1}{4}\left[\int_0^1\{(\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+(\zeta((1-t)\mu+mt\nu)\xi((1-t)\mu+mt\nu)\}dt\right]\\
&=\frac{1}{4}\left[\left(2\zeta(\mu)\xi(\mu)+2m^2\zeta(\nu)\xi(\nu)\right)\int_0^1 t^{\alpha}(1-t^{\alpha})dt+m(\zeta(\mu)\xi(\nu)\right]\\
&+\frac{\zeta(\nu)\xi(\mu)}{4}\int_0^1\left(t^{2\alpha}+(1-t^{\alpha})^2\right)dt\\
&=\frac{1}{4}\left[\int_0^1\{(\zeta(t\mu+t_m\nu)(\xi(t\mu+t_m\nu))+(\zeta(1-t)\mu+mt\nu)\xi(1-t)\mu+mt\nu)\}dt\right]\\
&+\frac{m^2\alpha(\zeta(\mu)\xi(\mu)+\zeta(\nu)\xi(\nu))}{2(\alpha+1)(2\alpha+1))}+\frac{m\left(2\alpha^2+\alpha+1\right)(\zeta(\mu)\xi(\nu)+\xi(\mu)\zeta(\nu))}{4(\alpha+1)(2\alpha+1)}.
\end{aligned} \tag{39}$$

Multiplying by 2 using the inequality (22), one can obtain the required inequality (34). □

Remark 3. By putting $\alpha=1$ and $m=1$, we obtain the first and second inequalities of Theorem 6.

Theorem 11. *Let ζ,ξ: $[0,\infty)\longrightarrow\mathbb{R}$ be a (α,m)-convex function. If $0\le\mu<\nu<\infty$, and $\zeta,\xi\in L_1[\mu,\nu]$, then one has the inequalities:*

$$\begin{aligned}
&\frac{3}{2(\nu-\mu)^2}\int_{\mu}^{\nu}\int_{\mu}^{\nu}\int_0^1\zeta(tx+t_my)\xi(tx+t_my)dtdxdy,\\
&\le\frac{3\left((\alpha+1)+2m^2\alpha^2\right)}{2(2\alpha+1)(\alpha+1)(\nu-\mu)}\int_{\mu}^{\nu}\zeta(x)\xi(x)dx+\frac{3m\alpha(M(\mu,\nu)+N(\mu,\nu))}{4(\alpha+1)(2\alpha+1)(\nu-\mu)^2},
\end{aligned} \tag{40}$$

and

$$\frac{3}{\nu-\mu}\int_\mu^\nu\int_0^1\zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)dtdx$$
$$\leq\frac{3}{(\nu-\mu)(2\alpha+1)}\int_\mu^\nu\zeta(x)\xi(x)dx+\frac{3m\alpha(1+m\alpha(\nu-\mu))(M(\mu,\nu)+N(\mu,\nu))}{2(2\alpha+1)(\alpha+1)(\nu-\mu)}, \tag{41}$$

where $M(\mu,\nu)$ and $N(\mu,\nu)$ are defined in Theorem 8.

Proof 4. Since ζ and ξ are (α,m)-convex on $[x,y]$, for $x,y\in[\mu,\nu]$ and $t\in[0,1]$, one can have

$$\zeta(tx+t_my).\xi(tx+t_my)$$
$$\leq t^{2\alpha}\zeta(x)\xi(x)+m^2(1-t^\alpha)^2\zeta(y)\xi(y)+mt^\alpha(1-t^\alpha)(\zeta(x)\xi(y)+\zeta(y)\xi(x)). \tag{42}$$

The above inequality in the integral form is as follows:

$$\int_0^1\zeta(tx+t_my)\xi(tx+t_my)dt$$
$$\leq\frac{\zeta(x)\xi(x)}{2\alpha+1}+\frac{2\alpha^2m^2\zeta(y)\xi(y)}{(2\alpha+1)(\alpha+1)}+\frac{m\alpha(\zeta(x)\xi(y)+\xi(x)\zeta(y))}{(2\alpha+1)(\alpha+1)}. \tag{43}$$

Integrating both sides on $[\mu,\nu]\times[\mu,\nu]$, one can obtain

$$\int_\mu^\nu\int_\mu^\nu\int_0^1\zeta(t\mu+t_m\nu)\xi(t\mu+t_m\nu)dtd\nu d\mu\leq\frac{\nu-\mu}{2\alpha+1}\int_\mu^\nu\zeta(x)\xi(x)dx$$
$$+\frac{2m^2\alpha^2(\nu-\mu)}{(2\alpha+1)(\alpha+1)}\int_\mu^\nu\zeta(y)\xi(y)dy+\frac{2m\alpha\int_\mu^\nu\zeta(x)dx\int_\mu^\nu\xi(y)dy}{(2\alpha+1)(\alpha+1)},$$
$$\leq\frac{(\nu-\mu)\left((\alpha+1)+2m^2\alpha^2\right)}{(2\alpha+1)(\alpha+1)}\int_\mu^\nu\zeta(x)\xi(x)dx+\frac{m\alpha(\zeta(\mu)+\zeta(\nu))(\xi(\mu)+\xi(\nu))}{2(\alpha+1)(2\alpha+1)},$$
$$=\frac{(\nu-\mu)\left((\alpha+1)+2m^2\alpha^2\right)}{(2\alpha+1)(\alpha+1)}\int_\mu^\nu\zeta(x)\xi(x)dx+\frac{m\alpha(M(\mu,\nu)+N(\mu,\nu))}{2(\alpha+1)(2\alpha+1)}. \tag{44}$$

By multiplying both sides by $(3/2)/1(\nu-\mu)^2$, we obtain the required inequality (41). Since ζ and ξ are (α,m)-convex on $[\mu,\nu]$, we have

$$\zeta\left[tx+t_m\left(\frac{\mu+\nu}{2}\right)\right]\leq t^\alpha\zeta(x)+t_m^\alpha\zeta\left(\frac{\mu+\nu}{2}\right), \tag{45}$$

$$\xi\left[tx+t_m\left(\frac{\mu+\nu}{2}\right)\right]\leq t^\alpha\xi(x)+t_m^\alpha\xi\left(\frac{\mu+\nu}{2}\right), \tag{46}$$

from (47) and (2.16), we have

$$\zeta\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx+t_m\left(\frac{\mu+\nu}{2}\right)\right)\leq t^{2\alpha}\zeta(x)\xi(x)+(1-t^\alpha)^2\zeta\left(\frac{\mu+\nu}{2}\right)$$
$$\xi\left(\frac{\mu+\nu}{2}\right)+mt^\alpha(1-t^\alpha)\left(\zeta(x)\xi\left(\frac{\mu+\nu}{2}\right)+\zeta\left(\frac{\mu+\nu}{2}\right)\xi(x)\right). \tag{47}$$

Integrating over $[0,1]$, we obtain

TABLE 1 | Numerical verification for $\zeta(x) = x^2$, $\xi(x) = x^3$, $\mu = 1$, $\nu = 2$, and $m = 0.8$.

Estimate	Left-hand side	Right-hand side	Gap
Endpoint estimate (15)	4.3826	8.7600	4.3774
Midpoint estimate (16)	7.4259	11.1626	3.7367
Double-integral estimate (23)	8.8175	13.1100	4.2925
Centered double-integral estimate (24)	16.4854	26.7000	10.2146

$$\int_0^1 \zeta\left(tx + t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx + t_m\left(\frac{\mu+\nu}{2}\right)\right)dt \leq \frac{\zeta(x)\nu(x)}{2\alpha+1} + \frac{2m^2\alpha^2}{(2\alpha+1)(\alpha+1)} \zeta\left(\frac{\mu+\nu}{2}\right)\xi\left(\frac{\mu+\nu}{2}\right) + \frac{m\alpha}{(2\alpha+1)(\alpha+1)}\left(\zeta(x)\xi\left(\frac{\mu+\nu}{2}\right) + \xi(x)\zeta\left(\frac{\mu+\nu}{2}\right)\right). \tag{48}$$

As explained in the proof of previously solved inequality given above, $\zeta.\xi$ is integrable on $[\mu, \nu]$. Now integrating both sides of the above inequality, both sides on $[\mu, \nu]$, using the right half of the Hermite–Hadamard–type inequalities given in the above and convexity of ζ and ξ, we observe that

$$\begin{aligned}\int_\mu^\nu\int_0^1 \zeta\left(tx + t_m\left(\frac{\mu+\nu}{2}\right)\right)\xi\left(tx + t_m\left(\frac{\mu+\nu}{2}\right)\right)dtdx &\leq \frac{1}{2\alpha+1}\int_\mu^\nu \zeta(x)\xi(x)dx \\ &+ \frac{m^2\alpha^2(\nu-\mu)}{2(2\alpha+1)(\alpha+1)}((\zeta(\mu)+\zeta(\nu)(\xi(\mu)+\xi(\nu)) + \frac{m\alpha}{(2\alpha+1)(\alpha+1)} \\ &\left[\left(\frac{\zeta(\mu)+\zeta(\nu)}{2}\right)\left(\frac{\xi(\mu)+\xi(\nu)}{2}\right) + \left(\frac{\xi(\mu)+\xi(\nu)}{2}\right)\left(\frac{\zeta(\mu)+\zeta(\nu)}{2}\right)\right], \\ &= \frac{1}{2\alpha+1}\int_\mu^\nu \zeta(x)\xi(x)dx + \frac{m^2\alpha^2(\nu-\mu)(M(\mu,\nu)+N(\mu,\nu))}{2(\alpha+1)(2\alpha+1)} \\ &+ \frac{m\alpha(M(\mu,\nu)+N(\mu,\nu))}{2(2\alpha+1)(\alpha+1)}.\end{aligned} \tag{49}$$

Now, multiplying both sides by $3/(\nu-\mu)$, we obtain the required inequality (46). □

Remark 4. By putting $m = 1$ and $\alpha = 1$, we obtain the first and second inequalities of Theorem 7.

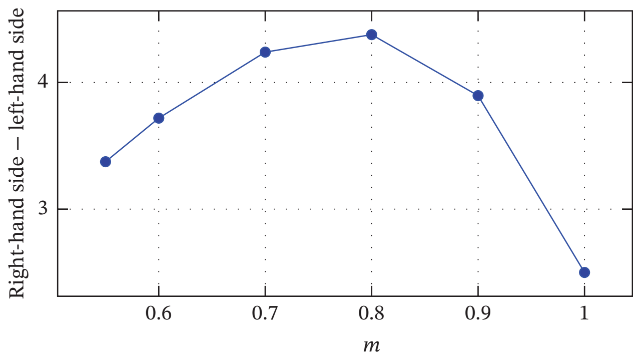

FIGURE 1 | Gap in the endpoint estimate (15) for $\zeta(x) = x^2$, $\xi(x) = x^3$, $\mu = 1$, and $\nu = 2$.

3 | Numerical Verification and Graphical Illustration

This section provides numerical checks for representative choices of the parameters. The computations are illustrative and are included to confirm the direction of the inequalities and the non-negativity of the resulting gaps. Let

$$\zeta(x) = x^2, \xi(x) = x^3, \mu = 1, \nu = 2, m = 0.8. \tag{50}$$

For these values, $m\nu = 1.6 > \mu$, so the normalized integral over $[\mu, m\nu]$ is well-defined. Table 1 lists the left-hand side, the right-hand side, and the gap for the four m-convex estimates.

Figure 1 shows the gap in the endpoint estimate (15) as m varies from 0.55 to 1.00, with the same functions and endpoints. The gap remains non-negative throughout the displayed interval, and at $m = 1$, the estimate coincides with the classical convex product estimate.

4 | Conclusion

In this article, we employ the product of functions to construct the class of generalized convex functions using two given functions. The focus of our study is on m-convex and

(α, m)-convex functions. After this, we utilize this approach to explore various types of Hermite–Hadamard–type inequalities. It has been observed that when " m" and " α" take on specific values, the outcomes meet with those of the product of m-convex and simple convex functions. When we use $\alpha = 1$, the results coincide with the product of m-convex functions. Moreover, our results align with the product of simple convex functions with $m = 1$. The comparison indicates that the derived results are improved and generalized the results obtained by Pachpatte in [33] in a distinctive manner. For those interested in delving deeper into this topic, one could investigate this product for $s - convex$, (s, m)-convex, harmonically convex, logarithmic convex, invex, preinvex, m-preinvex, and many other types of convex functions.

Author Contributions

Conceptualization: Abdur Rehman, Hala H. Taha, Waqas Nazeer, and Jongsuk Ro. Validation: Abdur Rehman, Hala H. Taha, Waqas Nazeer, and Jongsuk Ro. Writing manuscript: Abdur Rehman and Waqas Nazeer. Editing and review manuscript: Abdur Rehman, Hala H. Taha, Waqas Nazeer, and Jongsuk Ro.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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