

Article

Lyapunov-Based Controller Using Nonlinear Observer for Planar Motors

Khac Huan Su ¹, Jaeyun Yim ², Wonhee Kim ^{3,*}  and Youngwoo Lee ^{1,*} ¹ Department of Electrical Engineering, Chonnam National University, Gwangju 61186, Korea; 216948@jnu.ac.kr² Department of Energy System Engineering, Chung-Ang University, Seoul 06974, Korea; wodbs6807@cau.ac.kr³ School of Energy System Engineering, Chung-Ang University, Seoul 06974, Korea

* Correspondence: whkim79@cau.ac.kr (W.K.); stork@jnu.ac.kr (Y.L.)

Abstract: In general, it is not easy work to design controllers and observers for high-order nonlinear systems. Planar motors that are applied to semiconductor wafer-stage processes have 14th-order nonlinear dynamics and require high resolution for position tracking. Thus, many sensors are required to achieve enhanced tracking performance because there are many state variables. To handle these problems, we developed a Lyapunov-based controller to improve the position tracking performance. Consequently, a nonlinear observer (NOB) was also developed to estimate all of the state variables including the position, the velocity, and the phase current using only position feedback. The closed-loop stability is proved through Lyapunov theory and the input-to-state stability (ISS) property. The proposed method was evaluated based on the simulation results and compared with the conventional proportional–integral–derivative (PID) control method to show the improvement in the position tracking performance.

Keywords: planar motors; feedback linearization; position control; Lyapunov methods; nonlinear observer

MSC: 37N35



Citation: Su, K.H.; Yim, J.; Kim, W.; Lee, Y. Lyapunov-Based Controller Using Nonlinear Observer for Planar Motors. *Mathematics* **2022**, *10*, 2177. <https://doi.org/10.3390/math10132177>

Academic Editor: Nicu Bizon

Received: 25 May 2022

Accepted: 20 June 2022

Published: 22 June 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Planar motors are widely used to address high-precision motion control requirements in semiconductor wafer-stage systems, precision machine tools, and other automated manufacturing applications. Planar motors consist of two-axis X-Y directions that generate four forces mounted symmetrically on the housing part [1]. Forcers X_1 and X_2 generate force in the X direction while forcers Y_1 and Y_2 generate force in the Y direction, as shown in Figure 1. The yaw of the motor is generated by asymmetries in the forcers and leads to a loss of synchronization between the motor and platen teeth. Each forcer is capable of a high position resolution and high-speed motion using open-loop microstepping control. However, these forcers can have step-out, a long settling time, and especially low robustness in the presence of noise within the current. Additionally, the motor has high-order nonlinear dynamics with complex nonlinear properties. Thus, the motor requires a highly accurate position control method.

Microstepping was researched by Durkos in 1974 and has been widely used in control applications to improve resolution and significantly increase motion stability [2–5]. Microstepping uses a desired position to generate the desired currents. All these desired currents provide the planar motors with position tracking performance, because the torque generated by the desired currents can move the actual position to the desired position. Therefore, in order to improve the position tracking performance of microstepping, a feedback controller, considering the electrical dynamics, is required to guarantee the convergence of the phase currents to the desired phase currents of the electrical dynamics in planar motors.

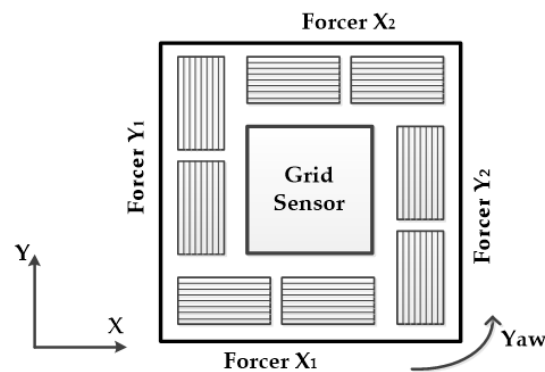


Figure 1. Top view of a planar motor.

Therefore, various studies have been developed to enhance the current tracking performance of planar motors [1,6–8]. A robust adaptive control with consideration of electrical dynamics is required to achieve the desired position performances [1]. A current controller with linear techniques based on nonlinear cascade control algorithms was proposed to enhance the transient response in the position control [6]. A nonlinear position control scheme using only position feedback under position tracking errors and yaw constraints was proposed to guarantee the tolerances for position tracking errors and yaw [7]. A linear parameter varying H_∞ control based on an augmented nonlinear observer was proposed to enhance both the position tracking performance and yaw regulation performance [8]. Similarly, a feedback controller considering electrical dynamics was required for improving the control performance of permanent magnet synchronous motors [9–11]. An adaptive internal model control and predictive current control with delay compensation were proposed to achieve a fast transient response, robustness, and higher-precision position tracking [9]. A novel active-disturbance-rejection-based sliding-mode current controller was studied to improve the transient current tracking performance and enhance robustness to internal disturbances [10]. A combination of sliding-mode predictive current control and an extended state observer was proposed to improve the control quality and robustness of the system [11]. Moreover, a current control scheme based on a conventional proportional–integral current control algorithm to adapt to changes in the parameters of permanent magnet stepper motors was studied to achieve improved position tracking performance [12]. In addition, applying the neural network–Kalman filter hybrid method to practice as an efficient method has allowed researchers to obtain state of charge estimations and aging state predictions for energy storage systems [13,14]. A novel recursive three-step filter was developed to simultaneously estimate the state and the unknown input with a rank-deficient distribution matrix [15]. Conventional controllers have been designed based on nonlinear dynamics, but it is still difficult to guarantee optimal tracking performance. Thus, it is highly necessary to create a linear system of nonlinear dynamics. Therefore, various approaches using other linearization techniques have been developed to address this requirement. A nonlinear system was transformed into a linear system by the input–output feedback linearization technique [16,17]. A robust feedback linearization technique for an induction motor was proposed [18]. The feedback linearization method based on direct torque control of interior permanent magnet synchronous motors was also investigated [19].

Since it is important to have the information of measured signals such as the position and current to linearize nonlinear dynamics, this can be obtained by implementing various sensors such as an encoder and a resolver; current feedback can be measured via sensing resistors. However, it is actually difficult to measure all state variables for high-order nonlinear dynamics because there are several problems such as cost and space limitations. To handle these problems, observer design is necessary to estimate all the state variables and to improve the transient response in position control.

In this article, a Lyapunov-based controller with a nonlinear observer (NOB) is proposed to enhance the position tracking performance. Microstepping uses the desired position to generate the desired currents. A Lyapunov-based controller using feedback linearization was developed to guarantee the exponential convergence of the phase currents to the desired phase currents for the electrical tracking error dynamics in the presence of back-electromotive forces and phase lag from the inductance. A nonlinear observer is proposed to estimate all the state variables (position, velocity, and phase currents) by using only the position feedback. The closed-loop stability is proved through Lyapunov theory and the input-to-state stability (ISS) property. The performance of the proposed control method is compared with the performance of the conventional proportional–integral–derivative (PID) control method based on the simulation results.

The main contributions of the proposed article can be summarized as follows:

1. **A Lyapunov-based controller using feedback linearization was designed to guarantee position tracking without linearization approximation errors.**
2. **A nonlinear observer was developed for state variable estimation to handle cost and space limitation problems.**
3. **Simulations were performed to improve the position tracking performance for both nominal parameters and parameter uncertainty.**

This article is organized into four sections. Section 2 describes the mathematical model of the planar motor and presents the stability analysis of the microstepping performance, and the design of the Lyapunov-based controller and nonlinear observer. Section 3 presents the simulation results of the proposed method. Finally, the conclusions are presented in Section 4.

2. Lyapunov-Based Controller and Nonlinear Observer Design based on a Mathematical Model

2.1. Mathematical Modeling of Planar Motor

The mathematical modeling of a planar motor consists of a mechanical part and an electrical part, which can be represented [1] as follows:

$$\begin{aligned}
 \dot{\theta}_x &= \omega_x \\
 \dot{\omega}_x &= \frac{-B_x \omega_x + F_{x1} + F_{x2}}{M} \\
 \dot{\theta}_y &= \omega_y \\
 \dot{\omega}_y &= \frac{-B_y \omega_y + F_{y1} + F_{y2}}{M} \\
 \dot{\theta}_{yaw} &= \omega_{yaw} \\
 \dot{\omega}_{yaw} &= \frac{-B_{yaw} \omega_{yaw} + I_x (F_{x1} - F_{x2}) + I_y (F_{y1} - F_{y2})}{J} \\
 \dot{i}_{x1a} &= \frac{-R i_{x1a} + \kappa \sin(\gamma \theta_{x1}) (\omega_x + I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x1a}}{L} \\
 \dot{i}_{x1b} &= \frac{-R i_{x1b} - \kappa \cos(\gamma \theta_{x1}) (\omega_x + I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x1b}}{L} \\
 \dot{i}_{x2a} &= \frac{-R i_{x2a} + \kappa \sin(\gamma \theta_{x2}) (\omega_x - I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x2a}}{L} \\
 \dot{i}_{x2b} &= \frac{-R i_{x2b} - \kappa \cos(\gamma \theta_{x2}) (\omega_x - I_x \cos(\theta_{yaw}) \omega_{yaw}) + v_{x2b}}{L} \\
 \dot{i}_{y1a} &= \frac{-R i_{y1a} + \kappa \sin(\gamma \theta_{y1}) (\omega_y + I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y1a}}{L} \\
 \dot{i}_{y1b} &= \frac{-R i_{y1b} - \kappa \cos(\gamma \theta_{y1}) (\omega_y + I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y1b}}{L} \\
 \dot{i}_{y2a} &= \frac{-R i_{y2a} + \kappa \sin(\gamma \theta_{y2}) (\omega_y - I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y2a}}{L} \\
 \dot{i}_{y2b} &= \frac{-R i_{y2b} - \kappa \cos(\gamma \theta_{y2}) (\omega_y - I_y \cos(\theta_{yaw}) \omega_{yaw}) + v_{y2b}}{L}
 \end{aligned} \tag{1}$$

where $\gamma = \frac{2\pi}{p}$ and

$$\begin{aligned} F_{x1} &= -\kappa \sin(\gamma\theta_{x1})i_{x1a} + \kappa \cos(\gamma\theta_{x1})i_{x1b}, F_{x2} = -\kappa \sin(\gamma\theta_{x2})i_{x2a} + \kappa \cos(\gamma\theta_{x2})i_{x2b} \\ F_{y1} &= -\kappa \sin(\gamma\theta_{y1})i_{y1a} + \kappa \cos(\gamma\theta_{y1})i_{y1b}, F_{y2} = -\kappa \sin(\gamma\theta_{y2})i_{y2a} + \kappa \cos(\gamma\theta_{y2})i_{y2b} \\ \theta_{x1} &= \theta_x + l_x \sin(\theta_{yaw}), \theta_{x2} = \theta_x - l_x \sin(\theta_{yaw}) \\ \theta_{y1} &= \theta_y + l_y \sin(\theta_{yaw}), \theta_{y2} = \theta_y - l_y \sin(\theta_{yaw}) \end{aligned}$$

where θ_x , θ_y , and θ_{yaw} are the positions of the X-axis and the Y-axis, and the yaw rotation of the motor, respectively; ω_x , ω_y , and ω_{yaw} are the velocities of the X-axis and the Y-axis, and the yaw rate of the motor, respectively; l_x and l_y are the distances from the center of the motor to the forcer; F_{x1} and F_{x2} are the forces from the two X-axis forcers, whereas F_{y1} and F_{y2} are the forces from the two Y-axis forcers; κ is the force constant; M and J denote the motor mass and inertia, respectively; B_x , B_y , and B_{yaw} are the coefficients of friction; p is the tooth pitch of the platen; i_{x1a} , i_{x1b} , i_{x2a} , i_{x2b} , i_{y1a} , i_{y1b} , i_{y2a} , and i_{y2b} are the phase currents; v_{x1a} , v_{x1b} , v_{x2a} , v_{x2b} , v_{y1a} , v_{y1b} , v_{y2a} , and v_{y2b} are the input voltages in phases A and B.

2.2. Stability Analysis of Microstepping Performance

In this section, we analyze the stability of the microstepping performance. We assume that all forces are perfectly synchronized and aligned well, so yaw and disturbances will not occur. Therefore, we assume that the current, velocity, and the position of X_1 and X_2 are equal, and those of Y_1 and Y_2 are also equal.

Theorem 1. To achieve the static desired position θ_x^d, θ_y^d for the planar motor, the desired microstepping inputs are given to dynamics (1) as follows:

$$\begin{aligned} v_{x1a}^d &= V_{\max} \cos(\gamma\theta_x^d), & v_{x1b}^d &= V_{\max} \sin(\gamma\theta_x^d) \\ v_{x2a}^d &= V_{\max} \cos(\gamma\theta_x^d), & v_{x2b}^d &= V_{\max} \sin(\gamma\theta_x^d) \\ v_{y1a}^d &= V_{\max} \cos(\gamma\theta_y^d), & v_{y1b}^d &= V_{\max} \sin(\gamma\theta_y^d) \\ v_{y2a}^d &= V_{\max} \cos(\gamma\theta_y^d), & v_{y2b}^d &= V_{\max} \sin(\gamma\theta_y^d) \end{aligned}$$

where $V_{\max}$ is the amplitude of the input voltage. Given $v_{x1a}^d, v_{x1b}^d, v_{x2a}^d, v_{x2b}^d, v_{y1a}^d, v_{y1b}^d, v_{y2a}^d$, and v_{y2b}^d in phases A and B, the states of the planar motor (1) locally asymptotically converge to an equilibrium point $x_e = [\theta_x^d, 0, \theta_y^d, 0, i_{x1a}^d, i_{x1b}^d, i_{x2a}^d, i_{x2b}^d, i_{y1a}^d, i_{y1b}^d, i_{y2a}^d, i_{y2b}^d]$, i.e.,

$$\begin{aligned} \lim_{t \rightarrow \infty} \theta_x(t) &= \theta_x^d, & \lim_{t \rightarrow \infty} \omega_x(t) &= 0 \\ \lim_{t \rightarrow \infty} \theta_y(t) &= \theta_y^d, & \lim_{t \rightarrow \infty} \omega_y(t) &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} i_{x1a}(t) &= i_{x1a}^d, & \lim_{t \rightarrow \infty} i_{x1b}(t) &= i_{x1b}^d \\ \lim_{t \rightarrow \infty} i_{x2a}(t) &= i_{x2a}^d, & \lim_{t \rightarrow \infty} i_{x2b}(t) &= i_{x2b}^d \\ \lim_{t \rightarrow \infty} i_{y1a}(t) &= i_{y1a}^d, & \lim_{t \rightarrow \infty} i_{y1b}(t) &= i_{y1b}^d \\ \lim_{t \rightarrow \infty} i_{y2a}(t) &= i_{y2a}^d, & \lim_{t \rightarrow \infty} i_{y2b}(t) &= i_{y2b}^d \end{aligned}$$

where $i_{x1a}^d = \frac{v_{x1a}^d}{R}$, $i_{x1b}^d = \frac{v_{x1b}^d}{R}$, $i_{x2a}^d = \frac{v_{x2a}^d}{R}$, $i_{x2b}^d = \frac{v_{x2b}^d}{R}$, $i_{y1a}^d = \frac{v_{y1a}^d}{R}$, $i_{y1b}^d = \frac{v_{y1b}^d}{R}$, $i_{y2a}^d = \frac{v_{y2a}^d}{R}$, and $i_{y2b}^d = \frac{v_{y2b}^d}{R}$ are the desired currents in phases A and B for microstepping defined as follows:

$$\begin{aligned} i_{x1a}^d &= I^d \cos(\gamma\theta_x^d), & i_{x1b}^d &= I^d \sin(\gamma\theta_x^d) \\ i_{x2a}^d &= I^d \cos(\gamma\theta_x^d), & i_{x2b}^d &= I^d \sin(\gamma\theta_x^d) \\ i_{y1a}^d &= I^d \cos(\gamma\theta_y^d), & i_{y1b}^d &= I^d \sin(\gamma\theta_y^d) \\ i_{y2a}^d &= I^d \cos(\gamma\theta_y^d), & i_{y2b}^d &= I^d \sin(\gamma\theta_y^d) \end{aligned} \quad (2)$$

where $I^d = \frac{V_{\max}}{R}$.

Proof of Theorem 1. The equilibrium points depend on the inputs; here, we only analyze the case $v_{x1a} = u_{x1a}$ and $v_{x1b} = u_{x1b}$. In this case, the dynamics of X_1 are as follows:

$$\begin{aligned} \dot{\theta}_{x1} &= \omega_{x1} \\ \dot{\omega}_{x1} &= \frac{1}{M} [-2\kappa \sin(\gamma\theta_{x1}) i_{x1a} + 2\kappa \cos(\gamma\theta_{x1}) i_{x1b} - B_x \omega_{x1}] \\ \dot{i}_{x1a} &= \frac{1}{L} [-R i_{x1a} + \kappa \omega_{x1} \sin(\gamma\theta_{x1}) + v_{x1a}] \\ \dot{i}_{x1b} &= \frac{1}{L} [-R i_{x1b} - \kappa \omega_{x1} \cos(\gamma\theta_{x1}) + v_{x1b}] \end{aligned} \quad (3)$$

Letting $\bar{i}_{x1a} = i_{x1a} - \frac{u_{x1a}}{R}$ and $\bar{i}_{x1b} = i_{x1b} - \frac{u_{x1b}}{R}$, the model (3) is rewritten as follows:

$$\begin{aligned} \dot{\theta}_{x1} &= \omega_{x1} \\ \dot{\omega}_{x1} &= \frac{1}{M} [-2\kappa (\bar{i}_{x1a} + \frac{u_{x1a}}{R}) \sin(\gamma\theta_{x1}) + 2\kappa (\bar{i}_{x1b} + \frac{u_{x1b}}{R}) \cos(\gamma\theta_{x1})] \\ \dot{i}_{x1a} &= \frac{1}{L} [-R \bar{i}_{x1a} + \kappa \omega_{x1} \sin(\gamma\theta_{x1})] \\ \dot{i}_{x1b} &= \frac{1}{L} [-R \bar{i}_{x1b} - \kappa \omega_{x1} \cos(\gamma\theta_{x1})] \end{aligned} \quad (4)$$

In this case, (4) has the equilibrium points:

$$[\theta_{x1}, \omega_{x1}, \bar{i}_{x1a}, \bar{i}_{x1b}] = [\theta_{ab}, 0, 0, 0] \quad (5)$$

where θ_{ab} satisfies that $u_{x1a} \sin(\gamma\theta_{ab}) = u_{x1b} \cos(\gamma\theta_{ab})$. To analyze the stability conditions of the equilibrium points (5), we consider a continuously differentiable function V_{ab} such that

$$\begin{aligned} V_{ab} &= \frac{1}{2} \left(\frac{M}{L} \omega_{x1}^2 + \bar{i}_{x1a}^2 + \bar{i}_{x1b}^2 \right) + \frac{\kappa u_{x1a}}{\gamma LR} (2\text{sgn}(u_{x1a}) - \cos(\gamma\theta_{x1})) \\ &\quad + \frac{\kappa u_{x1b}}{\gamma LR} (2\text{sgn}(u_{x1b}) - \sin(\gamma\theta_{x1})) \geq 0 \end{aligned} \quad (6)$$

Its time derivative along the trajectory (6) is given by

$$\dot{V}_{ab} = -\frac{B_x}{L} \omega_{x1}^2 - \frac{R}{L} \bar{i}_{x1a}^2 - \frac{R}{L} \bar{i}_{x1b}^2 \leq 0$$

According to Lasalle's theorem [20], the equilibrium points $[\theta_{x1}, \omega_{x1}, \bar{i}_{x1a}, \bar{i}_{x1b}] = [\theta_{ab}, 0, 0, 0]$ are locally asymptotically stable. $\square$

The equilibrium points of the other cases including $(v_{x1a} = u_{x1a}, v_{x1b} = 0)$ and $(v_{x1a} = 0, v_{x1b} = u_{x1b})$ can be analyzed in the same way.

2.3. Lyapunov-Based Controller Using Feedback Linearization

Differentiating the current tracking errors of the electrical dynamics [8] with respect to time, the current tracking error dynamics are given as follows:

$$\begin{aligned}
 \dot{e}_{x1a} &= \frac{1}{L} \left(\dot{L}i_{x1a}^d + Ri_{x1a} - \kappa \sin(\gamma\theta_{x1}) (\omega_x + l_x \cos(\theta_{yaw}) \omega_{yaw}) - v_{x1a} \right) \\
 \dot{e}_{x1b} &= \frac{1}{L} \left(\dot{L}i_{x1b}^d + Ri_{x1b} + \kappa \cos(\gamma\theta_{x1}) (\omega_x + l_x \cos(\theta_{yaw}) \omega_{yaw}) - v_{x1b} \right) \\
 \dot{e}_{x2a} &= \frac{1}{L} \left(\dot{L}i_{x2a}^d + Ri_{x2a} - \kappa \sin(\gamma\theta_{x2}) (\omega_x - l_x \cos(\theta_{yaw}) \omega_{yaw}) - v_{x2a} \right) \\
 \dot{e}_{x2b} &= \frac{1}{L} \left(\dot{L}i_{x2b}^d + Ri_{x2b} + \kappa \cos(\gamma\theta_{x2}) (\omega_x - l_x \cos(\theta_{yaw}) \omega_{yaw}) - v_{x2b} \right) \\
 \dot{e}_{y1a} &= \frac{1}{L} \left(\dot{L}i_{y1a}^d + Ri_{y1a} - \kappa \sin(\gamma\theta_{y1}) (\omega_y + l_y \cos(\theta_{yaw}) \omega_{yaw}) - v_{y1a} \right) \\
 \dot{e}_{y1b} &= \frac{1}{L} \left(\dot{L}i_{y1b}^d + Ri_{y1b} + \kappa \cos(\gamma\theta_{y1}) (\omega_y + l_y \cos(\theta_{yaw}) \omega_{yaw}) - v_{y1b} \right) \\
 \dot{e}_{y2a} &= \frac{1}{L} \left(\dot{L}i_{y2a}^d + Ri_{y2a} - \kappa \sin(\gamma\theta_{y2}) (\omega_y - l_y \cos(\theta_{yaw}) \omega_{yaw}) - v_{y2a} \right) \\
 \dot{e}_{y2b} &= \frac{1}{L} \left(\dot{L}i_{y2b}^d + Ri_{y2b} + \kappa \cos(\gamma\theta_{y2}) (\omega_y - l_y \cos(\theta_{yaw}) \omega_{yaw}) - v_{y2b} \right)
 \end{aligned} \tag{7}$$

Theorem 2. According to the current tracking error dynamics (7), the voltage inputs of the planar motor with the integral terms of the current tracking errors are defined as follows:

$$\begin{aligned}
 e_{x1az} &= \int_0^t e_{x1a} d\tau, & e_{x1bz} &= \int_0^t e_{x1b} d\tau, & e_{x2az} &= \int_0^t e_{x2a} d\tau, & e_{x2bz} &= \int_0^t e_{x2b} d\tau \\
 e_{y1az} &= \int_0^t e_{y1a} d\tau, & e_{y1bz} &= \int_0^t e_{y1b} d\tau, & e_{y2az} &= \int_0^t e_{y2a} d\tau, & e_{y2bz} &= \int_0^t e_{y2b} d\tau
 \end{aligned}$$

and

$$\begin{aligned}
 v_{x1a} &= \dot{L}i_{x1a}^d + Ri_{x1a} - \kappa \omega_x \sin(\gamma\theta_{x1}) - \kappa l_x \cos(\theta_{yaw}) \sin(\gamma\theta_{x1}) \omega_{yaw} + k_P e_{x1a} + k_I e_{x1az} \\
 v_{x1b} &= \dot{L}i_{x1b}^d + Ri_{x1b} + \kappa \omega_x \cos(\gamma\theta_{x1}) + \kappa l_x \cos(\theta_{yaw}) \cos(\gamma\theta_{x1}) \omega_{yaw} + k_P e_{x1b} + k_I e_{x1bz} \\
 v_{x2a} &= \dot{L}i_{x2a}^d + Ri_{x2a} - \kappa \omega_x \sin(\gamma\theta_{x2}) + \kappa l_x \cos(\theta_{yaw}) \sin(\gamma\theta_{x2}) \omega_{yaw} + k_P e_{x2a} + k_I e_{x2az} \\
 v_{x2b} &= \dot{L}i_{x2b}^d + Ri_{x2b} + \kappa \omega_x \cos(\gamma\theta_{x2}) - \kappa l_x \cos(\theta_{yaw}) \cos(\gamma\theta_{x2}) \omega_{yaw} + k_P e_{x2b} + k_I e_{x2bz} \\
 v_{y1a} &= \dot{L}i_{y1a}^d + Ri_{y1a} - \kappa \omega_y \sin(\gamma\theta_{y1}) - \kappa l_y \cos(\theta_{yaw}) \sin(\gamma\theta_{y1}) \omega_{yaw} + k_P e_{y1a} + k_I e_{y1az} \\
 v_{y1b} &= \dot{L}i_{y1b}^d + Ri_{y1b} + \kappa \omega_y \cos(\gamma\theta_{y1}) + \kappa l_y \cos(\theta_{yaw}) \cos(\gamma\theta_{y1}) \omega_{yaw} + k_P e_{y1b} + k_I e_{y1bz} \\
 v_{y2a} &= \dot{L}i_{y2a}^d + Ri_{y2a} - \kappa \omega_y \sin(\gamma\theta_{y2}) + \kappa l_y \cos(\theta_{yaw}) \sin(\gamma\theta_{y2}) \omega_{yaw} + k_P e_{y2a} + k_I e_{y2az} \\
 v_{y2b} &= \dot{L}i_{y2b}^d + Ri_{y2b} + \kappa \omega_y \cos(\gamma\theta_{y2}) - \kappa l_y \cos(\theta_{yaw}) \cos(\gamma\theta_{y2}) \omega_{yaw} + k_P e_{y2b} + k_I e_{y2bz}
 \end{aligned} \tag{8}$$

where k_P and k_I are the positive controller gains.

Proof of Theorem 2. Let us define the V Lyapunov candidate function as follows:

$$V = \frac{1}{2}e_{x1a}^2 + \frac{1}{2}k_I e_{x1a}^2 + \frac{1}{2}e_{x1b}^2 + \frac{1}{2}k_I e_{x1b}^2 + \frac{1}{2}e_{x2a}^2 + \frac{1}{2}k_I e_{x2a}^2 + \frac{1}{2}e_{x2b}^2 + \frac{1}{2}k_I e_{x2b}^2 \\ + \frac{1}{2}e_{y1a}^2 + \frac{1}{2}k_I e_{y1a}^2 + \frac{1}{2}e_{y1b}^2 + \frac{1}{2}k_I e_{y1b}^2 + \frac{1}{2}e_{y2a}^2 + \frac{1}{2}k_I e_{y2a}^2 + \frac{1}{2}e_{y2b}^2 + \frac{1}{2}k_I e_{y2b}^2$$

The derivative of V with respect to time is expressed as follows:

$$\dot{V} = e_{x1a}\dot{e}_{x1a} + k_I e_{x1a}\dot{e}_{x1a} + e_{x1b}\dot{e}_{x1b} + k_I e_{x1b}\dot{e}_{x1b} \\ + e_{x2a}\dot{e}_{x2a} + k_I e_{x2a}\dot{e}_{x2a} + e_{x2b}\dot{e}_{x2b} + k_I e_{x2b}\dot{e}_{x2b} \\ + e_{y1a}\dot{e}_{y1a} + k_I e_{y1a}\dot{e}_{y1a} + e_{y1b}\dot{e}_{y1b} + k_I e_{y1b}\dot{e}_{y1b} \\ + e_{y2a}\dot{e}_{y2a} + k_I e_{y2a}\dot{e}_{y2a} + e_{y2b}\dot{e}_{y2b} + k_I e_{y2b}\dot{e}_{y2b} \\ = -k_P e_{x1a}^2 - k_P e_{x1b}^2 - k_P e_{x2a}^2 - k_P e_{x2b}^2 \\ - k_P e_{y1a}^2 - k_P e_{y1b}^2 - k_P e_{y2a}^2 - k_P e_{y2b}^2$$

Therefore, the origin of the current tracking error dynamics (7) is exponentially stable. $\square$

2.4. Nonlinear Observer Design and Analysis of Closed-Loop System

In the previous section, we can assume that all the state variables are measurable. However, the measurement of all the state variables is extremely difficult using sensors because there are several problems such as cost and space limitations. This section describes the design of a nonlinear observer using only position feedback to obtain the knowledge of the state variables (position, velocity, current). The nonlinear observer is designed as follows:

$$\begin{aligned} \dot{\hat{\theta}}_x &= \hat{\omega}_x + l_{\theta x}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{\omega}}_x &= \frac{-B_x \hat{\omega}_x + \hat{F}_{x1} + \hat{F}_{x2}}{M} + l_{\omega x}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{\theta}}_y &= \hat{\omega}_y + l_{\theta y}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{\omega}}_y &= \frac{-B_y \hat{\omega}_y + \hat{F}_{y1} + \hat{F}_{y2}}{M} + l_{\omega y}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{\theta}}_{yaw} &= \hat{\omega}_{yaw} + l_{\theta yaw}(\theta_{yaw} - \hat{\theta}_{yaw}) \\ \dot{\hat{\omega}}_{yaw} &= \frac{-B_{yaw} \hat{\omega}_{yaw} + l_x(\hat{F}_{x1} - \hat{F}_{x2}) + l_y(\hat{F}_{y1} - \hat{F}_{y2})}{J} + l_{\omega yaw}(\theta_{yaw} - \hat{\theta}_{yaw}) \\ \dot{\hat{i}}_{x1a} &= \frac{-R\hat{i}_{x1a} + \kappa \sin(\gamma\theta_{x1})(\hat{\omega}_x + l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x1a}}{L} + l_{x1a}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x1b} &= \frac{-R\hat{i}_{x1b} - \kappa \cos(\gamma\theta_{x1})(\hat{\omega}_x + l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x1b}}{L} + l_{x1b}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x2a} &= \frac{-R\hat{i}_{x2a} + \kappa \sin(\gamma\theta_{x2})(\hat{\omega}_x - l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x2a}}{L} + l_{x2a}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{x2b} &= \frac{-R\hat{i}_{x2b} - \kappa \cos(\gamma\theta_{x2})(\hat{\omega}_x - l_x \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{x2b}}{L} + l_{x2b}(\theta_x - \hat{\theta}_x) \\ \dot{\hat{i}}_{y1a} &= \frac{-R\hat{i}_{y1a} + \kappa \sin(\gamma\theta_{y1})(\hat{\omega}_y + l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y1a}}{L} + l_{y1a}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y1b} &= \frac{-R\hat{i}_{y1b} - \kappa \cos(\gamma\theta_{y1})(\hat{\omega}_y + l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y1b}}{L} + l_{y1b}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y2a} &= \frac{-R\hat{i}_{y2a} + \kappa \sin(\gamma\theta_{y2})(\hat{\omega}_y - l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y2a}}{L} + l_{y2a}(\theta_y - \hat{\theta}_y) \\ \dot{\hat{i}}_{y2b} &= \frac{-R\hat{i}_{y2b} - \kappa \cos(\gamma\theta_{y2})(\hat{\omega}_y - l_y \cos(\theta_{yaw})\hat{\omega}_{yaw}) + v_{y2b}}{L} + l_{y2b}(\theta_y - \hat{\theta}_y) \end{aligned} \quad (9)$$

where

$$\begin{aligned}\hat{F}_{x1} &= -\kappa \sin(\gamma\theta_{x1})\hat{i}_{x1a} + \kappa \cos(\gamma\theta_{x1})\hat{i}_{x1b}, \hat{F}_{x2} = -\kappa \sin(\gamma\theta_{x2})\hat{i}_{x2a} + \kappa \cos(\gamma\theta_{x2})\hat{i}_{x2b} \\ \hat{F}_{y1} &= -\kappa \sin(\gamma\theta_{y1})\hat{i}_{y1a} + \kappa \cos(\gamma\theta_{y1})\hat{i}_{y1b}, \hat{F}_{y2} = -\kappa \sin(\gamma\theta_{y2})\hat{i}_{y2a} + \kappa \cos(\gamma\theta_{y2})\hat{i}_{y2b}\end{aligned}$$

The estimation error dynamics with the estimation errors [8] are given as follows:

$$\begin{aligned}\dot{\tilde{\theta}}_x &= \tilde{\omega}_x - l_{\theta x}\tilde{\theta}_x \\ \dot{\tilde{\omega}}_x &= \frac{-B_x\tilde{\omega}_x + \tilde{F}_{x1} + \tilde{F}_{x2}}{M} - l_{\omega x}\tilde{\theta}_x \\ \dot{\tilde{\theta}}_y &= \tilde{\omega}_y - l_{\theta y}\tilde{\theta}_y \\ \dot{\tilde{\omega}}_y &= \frac{-B_y\tilde{\omega}_y + \tilde{F}_{y1} + \tilde{F}_{y2}}{M} - l_{\omega y}\tilde{\theta}_y \\ \dot{\tilde{\theta}}_{yaw} &= \tilde{\omega}_{yaw} - l_{\theta yaw}\tilde{\theta}_{yaw} \\ \dot{\tilde{\omega}}_{yaw} &= \frac{-B_{yaw}\tilde{\omega}_{yaw} + l_x(\tilde{F}_{x1} - \tilde{F}_{x2}) + l_y(\tilde{F}_{y1} - \tilde{F}_{y2})}{J} - l_{\omega yaw}\tilde{\theta}_{yaw} \\ \dot{\tilde{i}}_{x1a} &= \frac{-R\tilde{i}_{x1a} + \kappa \sin(\gamma\theta_{x1})(\tilde{\omega}_x + l_x \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{x1a}\tilde{\theta}_x \\ \dot{\tilde{i}}_{x1b} &= \frac{-R\tilde{i}_{x1b} - \kappa \cos(\gamma\theta_{x1})(\tilde{\omega}_x + l_x \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{x1b}\tilde{\theta}_x \\ \dot{\tilde{i}}_{x2a} &= \frac{-R\tilde{i}_{x2a} + \kappa \sin(\gamma\theta_{x2})(\tilde{\omega}_x - l_x \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{x2a}\tilde{\theta}_x \\ \dot{\tilde{i}}_{x2b} &= \frac{-R\tilde{i}_{x2b} - \kappa \cos(\gamma\theta_{x2})(\tilde{\omega}_x - l_x \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{x2b}\tilde{\theta}_x \\ \dot{\tilde{i}}_{y1a} &= \frac{-R\tilde{i}_{y1a} + \kappa \sin(\gamma\theta_{y1})(\tilde{\omega}_y + l_y \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{y1a}\tilde{\theta}_y \\ \dot{\tilde{i}}_{y1b} &= \frac{-R\tilde{i}_{y1b} - \kappa \cos(\gamma\theta_{y1})(\tilde{\omega}_y + l_y \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{y1b}\tilde{\theta}_y \\ \dot{\tilde{i}}_{y2a} &= \frac{-R\tilde{i}_{y2a} + \kappa \sin(\gamma\theta_{y2})(\tilde{\omega}_y - l_y \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{y2a}\tilde{\theta}_y \\ \dot{\tilde{i}}_{y2b} &= \frac{-R\tilde{i}_{y2b} - \kappa \cos(\gamma\theta_{y2})(\tilde{\omega}_y - l_y \cos(\theta_{yaw})\tilde{\omega}_{yaw})}{L} - l_{y2b}\tilde{\theta}_y\end{aligned}\quad (10)$$

where

$$\begin{aligned}\tilde{F}_{x1} &= -\kappa \sin(\gamma\theta_{x1})\tilde{i}_{x1a} + \kappa \cos(\gamma\theta_{x1})\tilde{i}_{x1b}, \tilde{F}_{x2} = -\kappa \sin(\gamma\theta_{x2})\tilde{i}_{x2a} + \kappa \cos(\gamma\theta_{x2})\tilde{i}_{x2b} \\ \tilde{F}_{y1} &= -\kappa \sin(\gamma\theta_{y1})\tilde{i}_{y1a} + \kappa \cos(\gamma\theta_{y1})\tilde{i}_{y1b}, \tilde{F}_{y2} = -\kappa \sin(\gamma\theta_{y2})\tilde{i}_{y2a} + \kappa \cos(\gamma\theta_{y2})\tilde{i}_{y2b}\end{aligned}$$

Theorem 3. $l_{\theta x}, l_{\theta y}, l_{\theta yaw}, l_{\omega x}, l_{\omega y}, l_{\omega yaw}, l_{x1a}, l_{x1b}, l_{x2a}, l_{x2b}, l_{y1a}, l_{y1b}, l_{y2a},$ and l_{y2b} are estimator gains. If the estimator gains are chosen such that $l_{\theta x} > 0, l_{\theta y} > 0, l_{\theta yaw} > 0, l_{\omega x} = l_{\omega y} = \frac{L}{M}, l_{\omega yaw} = \frac{L}{J},$ and $l_{x1a}, l_{x1b}, l_{x2a}, l_{x2b}, l_{y1a}, l_{y1b}, l_{y2a},$ and l_{y2b} are zero, then the estimation errors exponentially converge to zero.

Proof of Theorem 3. Let us consider the V_{NOB} Lyapunov function candidate:

$$V_{NOB} = \frac{1}{2} \left(\tilde{\theta}_x^2 + \frac{M}{L}\tilde{\omega}_x^2 + \tilde{\theta}_y^2 + \frac{M}{L}\tilde{\omega}_y^2 + \tilde{\theta}_{yaw}^2 + \frac{J}{L}\cos(\theta_{yaw})\tilde{\omega}_{yaw}^2 + \tilde{i}_{x1a}^2 + \tilde{i}_{x1b}^2 + \tilde{i}_{x2a}^2 + \tilde{i}_{x2b}^2 + \tilde{i}_{y1a}^2 + \tilde{i}_{y1b}^2 + \tilde{i}_{y2a}^2 + \tilde{i}_{y2b}^2 \right)$$

We assume that the yaw is within $\pm 90^\circ$ so that V_{NOB} is always positive definite. The derivative of V_{NOB} with respect to time is expressed as follows:

$$\begin{aligned}\dot{V}_{NOB} = & -l_{\theta x}\tilde{\theta}_x^2 - \frac{B_x}{L}\tilde{\omega}_x^2 - l_{\theta y}\tilde{\theta}_y^2 - \frac{B_y}{L}\tilde{\omega}_y^2 - l_{\theta yaw}\tilde{\theta}_{yaw}^2 - \frac{B_{yaw}}{L}\tilde{\omega}_{yaw}^2 \\ & - \frac{R}{L}\tilde{i}_{x1a}^2 - \frac{R}{L}\tilde{i}_{x1b}^2 - \frac{R}{L}\tilde{i}_{x2a}^2 - \frac{R}{L}\tilde{i}_{x2b}^2 - \frac{R}{L}\tilde{i}_{y1a}^2 - \frac{R}{L}\tilde{i}_{y1b}^2 - \frac{R}{L}\tilde{i}_{y2a}^2 - \frac{R}{L}\tilde{i}_{y2b}^2\end{aligned}$$

Indeed, $\dot{V}_{NOB}$ is negative definite. Therefore, $\tilde{X}_{NOB}$ exponentially converges to zero. $\square$

Let us proceed to the analysis of the closed-loop system combined with the proposed controller (8) and NOB (9) with the estimated current tracking errors [8]; the voltage inputs of the planar motor and the current tracking error dynamics are represented as follows:

$$\begin{aligned}v_{x1a} &= \dot{L}i_{x1a}^d + R\hat{i}_{x1a} - \kappa\hat{\omega}_x \sin(\gamma\theta_{x1}) - \kappa l_x \cos(\theta_{yaw}) \sin(\gamma\theta_{x1})\hat{\omega}_{yaw} + k_P\hat{e}_{x1a} + k_I\hat{e}_{x1az} \\ v_{x1b} &= \dot{L}i_{x1b}^d + R\hat{i}_{x1b} + \kappa\hat{\omega}_x \cos(\gamma\theta_{x1}) + \kappa l_x \cos(\theta_{yaw}) \cos(\gamma\theta_{x1})\hat{\omega}_{yaw} + k_P\hat{e}_{x1b} + k_I\hat{e}_{x1bz} \\ v_{x2a} &= \dot{L}i_{x2a}^d + R\hat{i}_{x2a} - \kappa\hat{\omega}_x \sin(\gamma\theta_{x2}) + \kappa l_x \cos(\theta_{yaw}) \sin(\gamma\theta_{x2})\hat{\omega}_{yaw} + k_P\hat{e}_{x2a} + k_I\hat{e}_{x2az} \\ v_{x2b} &= \dot{L}i_{x2b}^d + R\hat{i}_{x2b} + \kappa\hat{\omega}_x \cos(\gamma\theta_{x2}) - \kappa l_x \cos(\theta_{yaw}) \cos(\gamma\theta_{x2})\hat{\omega}_{yaw} + k_P\hat{e}_{x2b} + k_I\hat{e}_{x2bz} \\ v_{y1a} &= \dot{L}i_{y1a}^d + R\hat{i}_{y1a} - \kappa\hat{\omega}_y \sin(\gamma\theta_{y1}) - \kappa l_y \cos(\theta_{yaw}) \sin(\gamma\theta_{y1})\hat{\omega}_{yaw} + k_P\hat{e}_{y1a} + k_I\hat{e}_{y1az} \\ v_{y1b} &= \dot{L}i_{y1b}^d + R\hat{i}_{y1b} + \kappa\hat{\omega}_y \cos(\gamma\theta_{y1}) + \kappa l_y \cos(\theta_{yaw}) \cos(\gamma\theta_{y1})\hat{\omega}_{yaw} + k_P\hat{e}_{y1b} + k_I\hat{e}_{y1bz} \\ v_{y2a} &= \dot{L}i_{y2a}^d + R\hat{i}_{y2a} - \kappa\hat{\omega}_y \sin(\gamma\theta_{y2}) + \kappa l_y \cos(\theta_{yaw}) \sin(\gamma\theta_{y2})\hat{\omega}_{yaw} + k_P\hat{e}_{y2a} + k_I\hat{e}_{y2az} \\ v_{y2b} &= \dot{L}i_{y2b}^d + R\hat{i}_{y2b} + \kappa\hat{\omega}_y \cos(\gamma\theta_{y2}) - \kappa l_y \cos(\theta_{yaw}) \cos(\gamma\theta_{y2})\hat{\omega}_{yaw} + k_P\hat{e}_{y2b} + k_I\hat{e}_{y2bz}\end{aligned}\quad (11)$$

and

$$\begin{aligned}\dot{e}_{x1a} &= \frac{1}{L} \left[-(k_P + k_I)e_{x1a} - (R + k_P + k_I)\tilde{i}_{x1a} + \kappa\tilde{\omega}_x \sin(\gamma\theta_{x1}) + \kappa l_x \sin(\gamma\theta_{x1}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right] \\ \dot{e}_{x1b} &= \frac{1}{L} \left(-(k_P + k_I)e_{x1b} - (R + k_P + k_I)\tilde{i}_{x1b} - \kappa\tilde{\omega}_x \cos(\gamma\theta_{x1}) - \kappa l_x \cos(\gamma\theta_{x1}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right) \\ \dot{e}_{x2a} &= \frac{1}{L} \left(-(k_P + k_I)e_{x2a} - (R + k_P + k_I)\tilde{i}_{x2a} + \kappa\tilde{\omega}_x \sin(\gamma\theta_{x2}) - \kappa l_x \sin(\gamma\theta_{x2}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right) \\ \dot{e}_{x2b} &= \frac{1}{L} \left(-(k_P + k_I)e_{x2b} - (R + k_P + k_I)\tilde{i}_{x2b} - \kappa\tilde{\omega}_x \cos(\gamma\theta_{x2}) + \kappa l_x \cos(\gamma\theta_{x2}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right) \\ \dot{e}_{y1a} &= \frac{1}{L} \left[-(k_P + k_I)e_{y1a} - (R + k_P + k_I)\tilde{i}_{y1a} + \kappa\tilde{\omega}_y \sin(\gamma\theta_{y1}) + \kappa l_y \sin(\gamma\theta_{y1}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right] \\ \dot{e}_{y1b} &= \frac{1}{L} \left(-(k_P + k_I)e_{y1b} - (R + k_P + k_I)\tilde{i}_{y1b} - \kappa\tilde{\omega}_y \cos(\gamma\theta_{y1}) - \kappa l_y \cos(\gamma\theta_{y1}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right) \\ \dot{e}_{y2a} &= \frac{1}{L} \left(-(k_P + k_I)e_{y2a} - (R + k_P + k_I)\tilde{i}_{y2a} + \kappa\tilde{\omega}_y \sin(\gamma\theta_{y2}) - \kappa l_y \sin(\gamma\theta_{y2}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right) \\ \dot{e}_{y2b} &= \frac{1}{L} \left(-(k_P + k_I)e_{y2b} - (R + k_P + k_I)\tilde{i}_{y2b} - \kappa\tilde{\omega}_y \cos(\gamma\theta_{y2}) + \kappa l_y \cos(\gamma\theta_{y2}) \cos(\theta_{yaw})\tilde{\omega}_{yaw} \right)\end{aligned}\quad (12)$$

From the estimation error dynamics (10) and current tracking error dynamics (12), the closed-loop dynamics are expressed as follows (refer to Appendix A):

$$\begin{aligned}\dot{e}_{elec} &= A_{elec}e_{elec} + B_{elec}\tilde{X}_{NOB} \\ \dot{\tilde{X}}_{NOB} &= A_{NOB}\tilde{X}_{NOB}\end{aligned}\quad (13)$$

where $e_{elec} = [e_{x1a}, e_{x1b}, e_{x2a}, e_{x2b}, e_{y1a}, e_{y1b}, e_{y2a}, e_{y2b}]^T$. In the closed-loop system (13), A_{elec} is Hurwitz with the positive control gains and B_{elec} is bounded. Thus, e_{elec} is ISS stable,

because $\tilde{X}_{NOB}$ was proven to exponentially converge to zero in the previous section [20]. Therefore, e_{elec} converges to zero.

The control block diagram of the proposed method is shown in Figure 2.

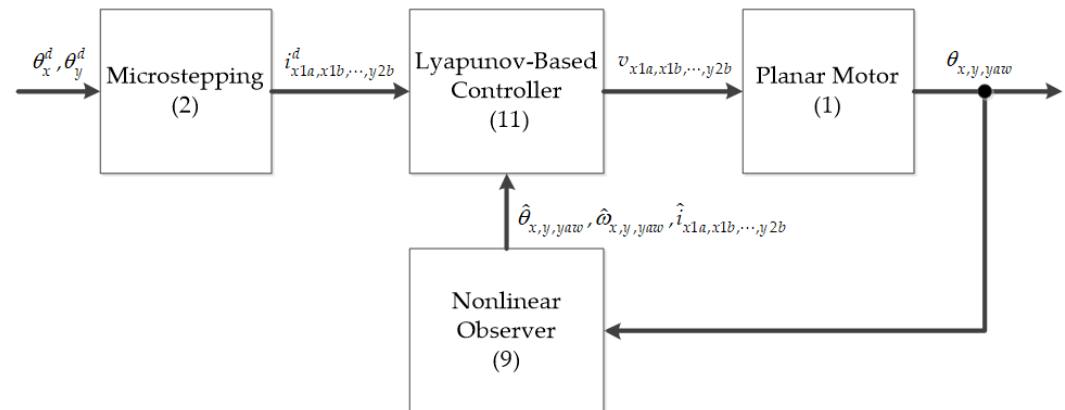


Figure 2. Control block diagram of the proposed algorithm.

3. Simulation Results

Simulations were performed to evaluate the performance of the proposed control method using MATLAB/Simulink. The planar motor parameters, control gains, and nonlinear observer gains are listed in Tables 1 and 2. The seventh-order profile was used for the reference position profiles shown in Figure 3, and the yaw desired profile was set to 0 rad.

Table 1. Parameters of the planar motor.

Parameters	Values
J	$4 \times 10^{-3} \text{ kg} \cdot \text{m}^2$
κ	17 N/A
p	$1.016 \times 10^{-3} \text{ m}$
R	2Ω
L	$7 \times 10^{-4} \text{ H}$
M	1.8 kg
l_x, l_y	0.0485 m
B_x, B_y, B_{yaw}	$1 \times 10^{-5} \text{ N} \cdot \text{m} \cdot \text{s/rad}$
γ	$2 \times \pi \times p$
$V_{\max}$	30 V

Table 2. Control gains and nonlinear observer gains.

Gains	Values
k_p	1
k_I	1000
$l_{\theta x}, l_{\theta y}$	5×10^3
$l_{\theta yaw}$	1×10^2
$l_{\omega x}, l_{\omega y}$	3.89×10^{-4}
$l_{\omega yaw}$	0.175
$l_{x1a}, l_{x1b}, l_{x2a}, l_{x2b}$	0
$l_{y1a}, l_{y1b}, l_{y2a}, l_{y2b}$	0

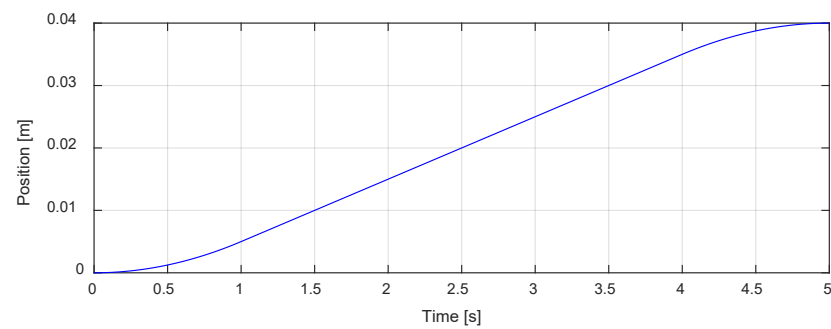
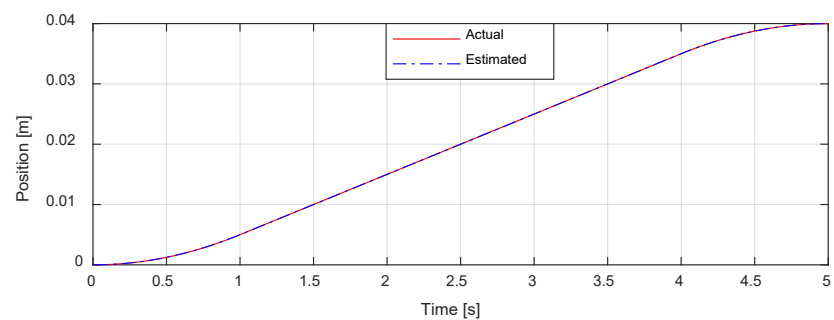


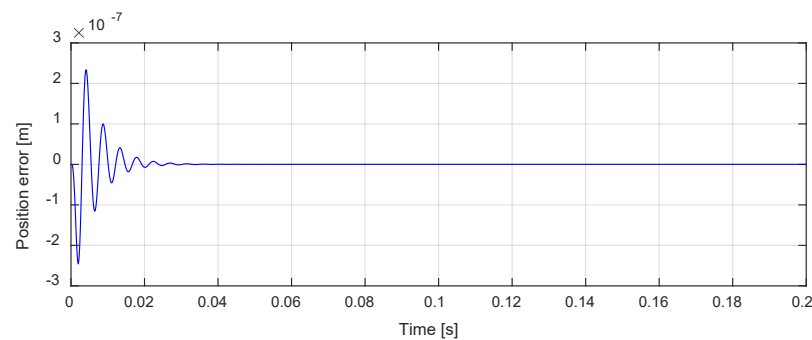
Figure 3. X-axis and Y-axis reference position profiles.

3.1. Simulation Results with Nominal Parameters

The estimation performances of the proposed NOB for the X-axis position, yaw rotation, and phase currents are shown in Figures 4–6, respectively. Because the proposed NOB was designed to estimate state variables, all of the estimation errors for the X-axis, yaw, and currents converged to zero in a steady-state region. However, transient errors still arose in an acceleration velocity region before 1 s.

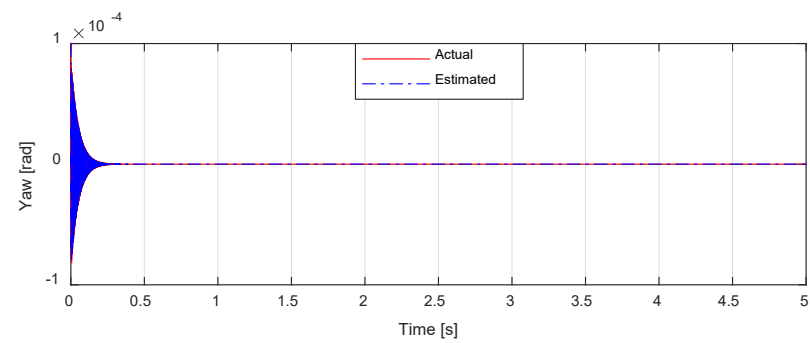


(a)

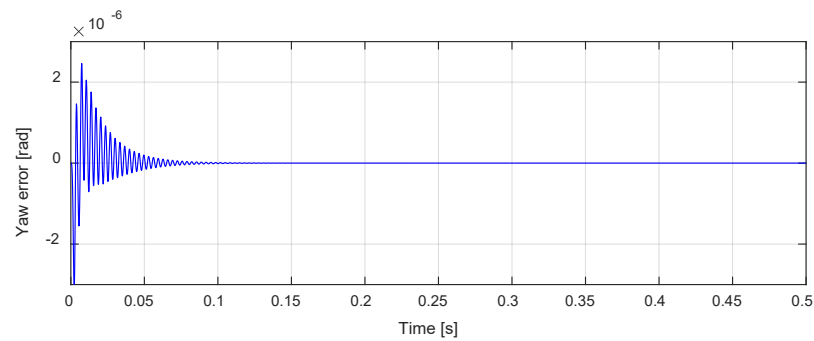


(b)

Figure 4. Estimation performance of X-axis. **(a)** Estimation performance of θ_x and $\hat{\theta}_x$. **(b)** $\tilde{\theta}_x$ estimation error.

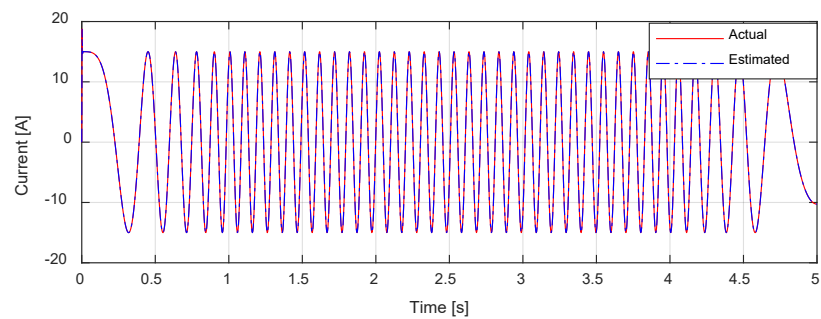


(a)

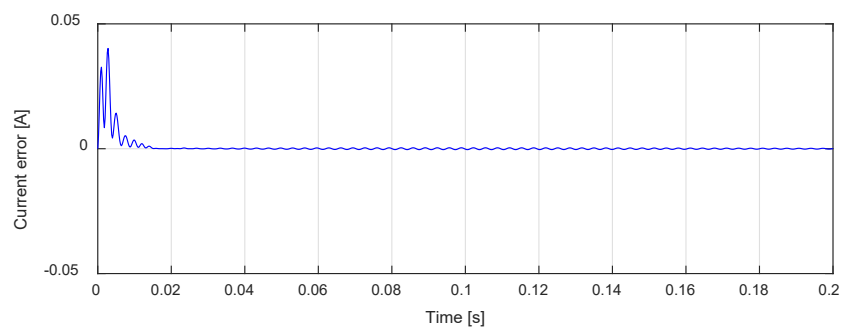


(b)

Figure 5. Estimation performance of yaw. (a) Estimation performance of θ_{yaw} and $\hat{\theta}_{yaw}$. (b) $\tilde{\theta}_{yaw}$ estimation error.



(a)



(b)

Figure 6. Estimation performance of phase A current. (a) Phase current estimation performance in phase A. (b) Phase current estimation error in phase A.

To validate the effectiveness of the proposed control method, we define the tracking errors of the mechanical part including $e_{\theta_x} = \theta_x^d - \theta_x$, $e_{\theta_y} = \theta_y^d - \theta_y$, and $e_{\theta_{yaw}} = \theta_{yaw}^d - \theta_{yaw}$ with the desired position of the X-axis and Y-axis and the yaw rotation as θ_x^d , θ_y^d , and θ_{yaw}^d , respectively. To evaluate the simulation results, there are two simulation cases:

Case 1: Conventional PID controller.

Case 2: Proposed Lyapunov-based controller.

In Case 1, the position and yaw tracking errors shown in Figure 7 had a relatively large overshoot during the acceleration and deceleration periods, and there were oscillations in the yaw motion. The large fixed controller gains resulted in a poor transient response in the position control. In contrast, the position and yaw tracking errors in Case 2 were improved, and the oscillation in the yaw motion was considerably reduced by the microstepping with the Lyapunov-based controller using feedback linearization. We confirm that the proposed control method improves the position tracking performance.

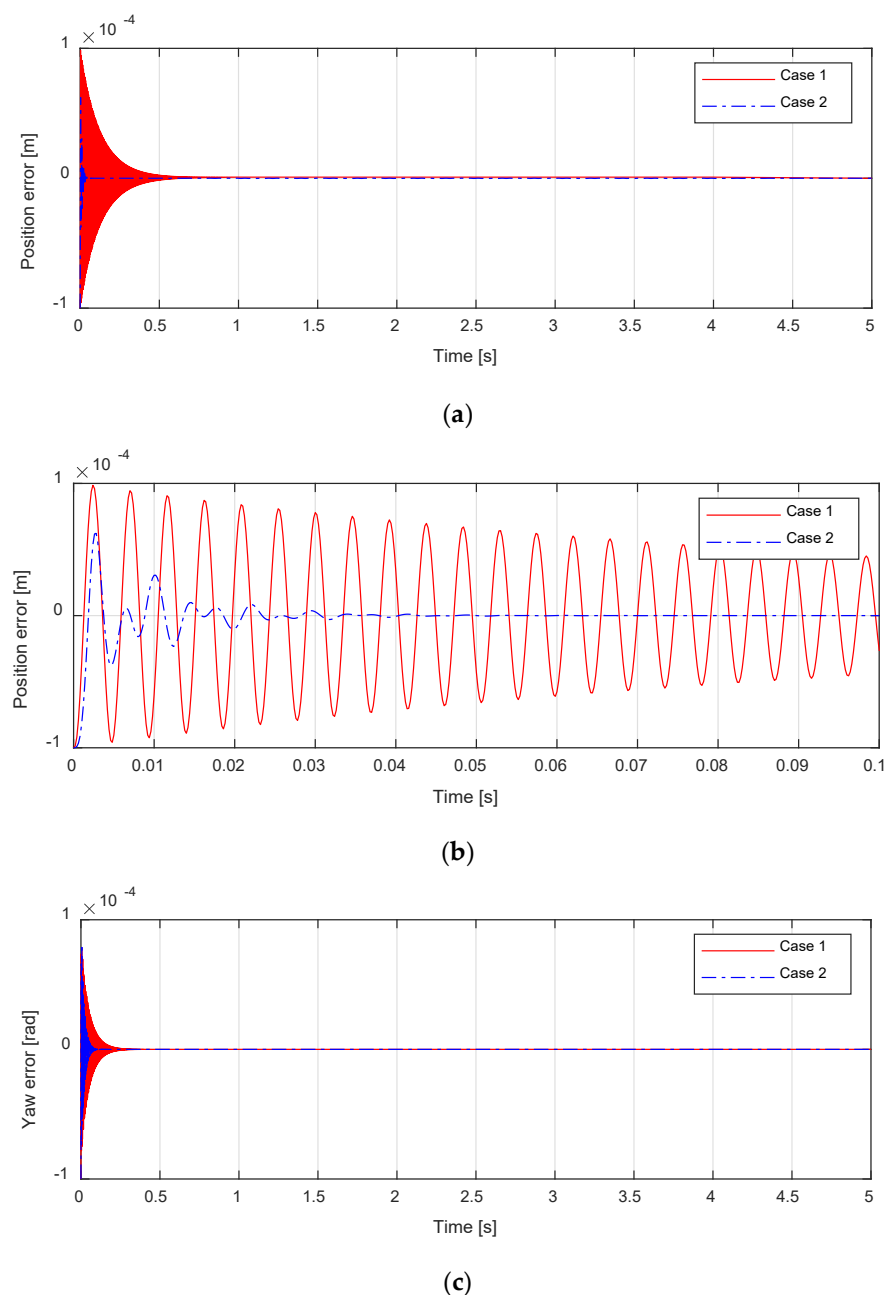
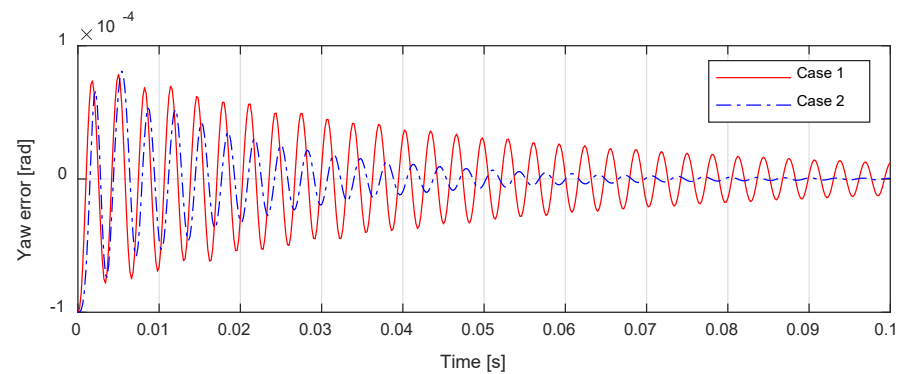


Figure 7. Cont.



(d)

Figure 7. Position tracking performance of X-axis position and yaw. (a) X-axis position tracking performance. (b) X-axis position tracking performance (enlarged). (c) Yaw tracking performance. (d) Yaw tracking performance (enlarged).

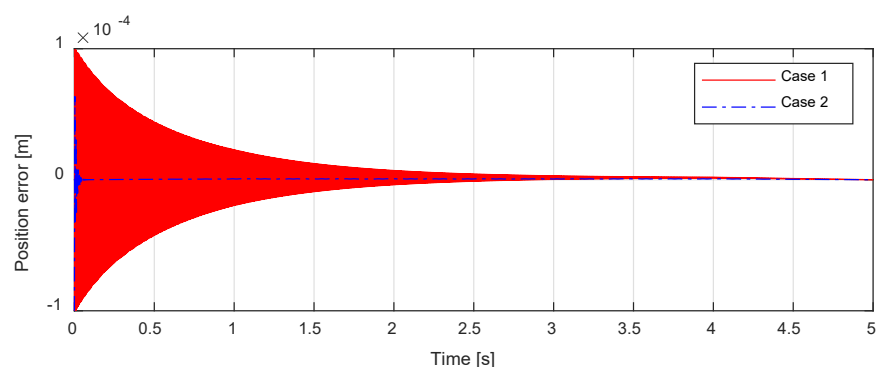
3.2. Simulation Results with Parameter Uncertainty

In this subsection, we present another simulation we performed to evaluate the robustness against parameter uncertainty. For electromechanical systems such as planar motors, plant parameters are generally not constant due to their mechanical structure and manufacturing tolerance. In particular, electrical parameters such as the resistor and inductor can be more easily changed depending on the environment (thermal condition and frequency). Therefore, let us define the parameter uncertainty of the resistor and inductor as follows:

$$R_r = R + \Delta R, \quad 0.9R \leq \Delta R \leq 1.1R$$

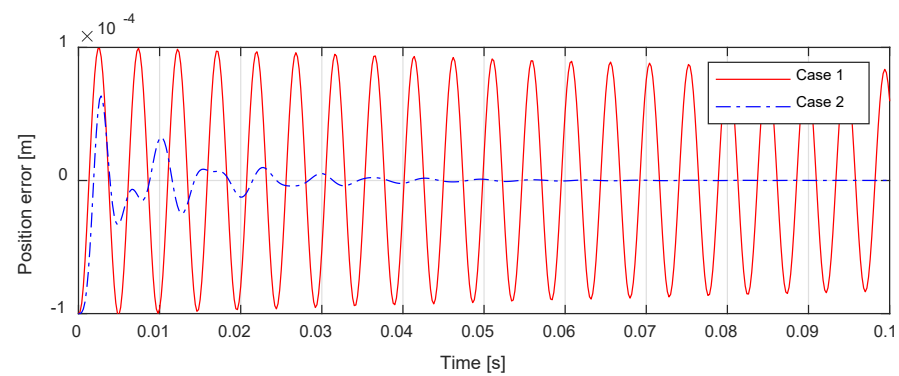
$$L_r = L + \Delta L, \quad 0.9L \leq \Delta L \leq 1.1L$$

where R_r and L_r are parameters considering uncertainties; ΔR and ΔL are the uncertainty of the resistor and inductor, respectively. For 10% of the nominal parameters, the position tracking performances of the X-axis and yaw are shown in Figure 8. Compared with the nominal parameter simulation results, Case 1 has a larger settling time. On the other hand, the amplitude of the tracking error and the settling time are similar to the nominal parameter case with the proposed method. This means that the proposed method has robustness against parameter uncertainty since it was designed by considering integral action.

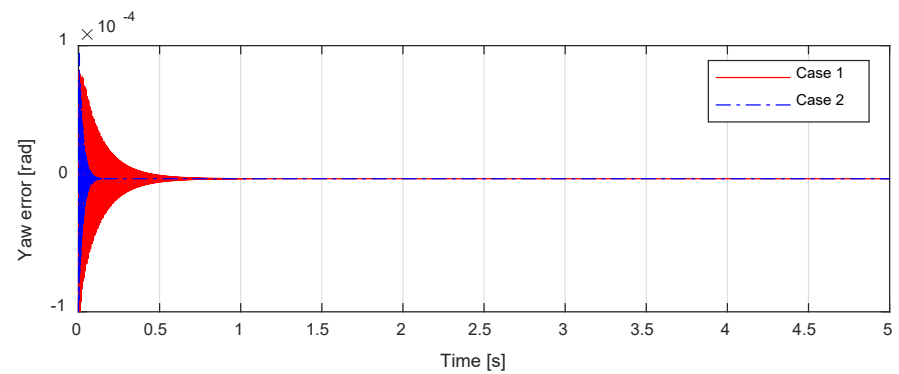


(a)

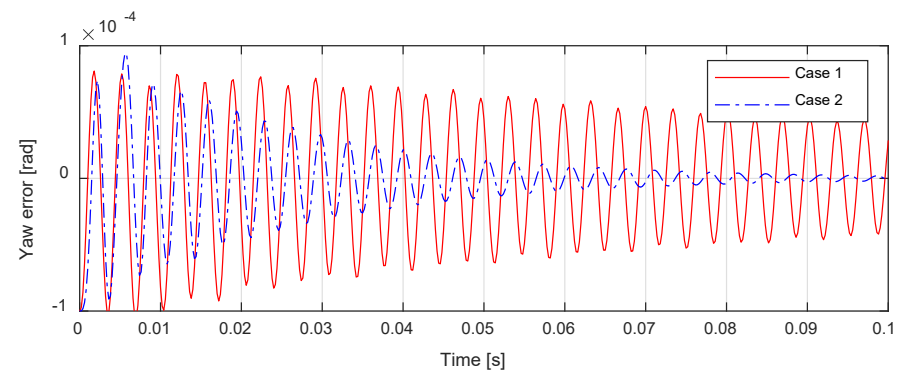
Figure 8. Cont.



(b)



(c)



(d)

Figure 8. Position tracking performance of X-axis position and yaw with uncertain parameters. (a) X-axis position tracking performance. (b) X-axis position tracking performance (enlarged). (c) Yaw tracking performance. (d) Yaw tracking performance (enlarged).

4. Conclusions

In this article, a Lyapunov-based controller using feedback linearization was designed to enhance the position tracking performance. Microstepping generated the desired phase currents to track the desired torque. To estimate the state variables, a NOB was developed using only position feedback. The closed-loop stability was proven by Lyapunov theory and the ISS property. Simulations were performed to evaluate the effectiveness of the proposed method by comparison with a PID controller. In the steady-state region, the position tracking errors of the X- and Y-axis and the yaw converged more rapidly to zero using the proposed method. Where parameter uncertainty existed, the proposed method had a similar transient response to the nominal parameter results. On the other hand,

the amplitude of the position tracking error and the settling time were increased in the transient region.

One possible future research direction is to consider disturbances such as load torque. Another possible future research direction would be to optimize the tracking performance by designing optimal controllers.

Author Contributions: K.H.S. and J.Y. designed the control algorithm and performed the simulation; Y.L. and W.K. developed the observer algorithm and revised the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This study was financially supported by Chonnam National University (Grant number: 2021-2198) and partly supported by the Chung-Ang University Research Scholarship Grants in 2021.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

Nomenclature

θ_x and θ_y	Positions of the X-axis and the Y-axis [m].
θ_{yaw}	Yaw rotation [rad].
ω_x and ω_y	Velocities of the X-axis and Y-axis [m/s].
ω_{yaw}	Yaw rate [rad/s].
l_x and l_y	Distances from the center of the motor to the forcer [m].
F_{x1} and F_{x2}	Forces from the two X-axis forcers [N].
F_{y1} and F_{y2}	Forces from the two Y-axis forcers [N].
κ	Force constant [N/A].
M	Mass of the motor [kg].
J	Inertia of the motor [$kg \cdot m^2$].
B_x , B_y , and B_{yaw}	Coefficients of friction [$N \cdot m \cdot s / rad$].
p	Tooth pitch of the platen [m].
i_{x1a} , i_{x1b} , i_{x2a} , i_{x2b} , i_{y1a} , i_{y1b} , i_{y2a} , and i_{y2b}	Currents of phases A and B [A].
v_{x1a} , v_{x1b} , v_{x2a} , v_{x2b} , v_{y1a} , v_{y1b} , v_{y2a} , and v_{y2b}	Voltages of phases A and B [V].
V_{max}	Amplitude of the input voltage [V].
k_p , k_I	Control gains.
l_{θ_x} , l_{θ_y} , $l_{\theta_{yaw}}$, l_{x1a} , l_{x1b} , l_{x2a} , l_{x2b} , l_{y1a} , l_{y1b} , l_{y2a} and l_{y2b}	Observer gains.

Appendix A

$$\begin{aligned}
A_{elec} &= -(k_P + k_I) \text{diag}[1, 1, 1, 1, 1, 1, 1, 1], B_{elec} = [B_{elec1}, B_{elec2}] \\
B_{elec1} &= [C^{(1)}, C^{(2)}, C^{(3)}, C^{(4)}, C^{(5)}, C^{(6)}], C^{(1)} = 0_{8,1} \\
C^{(2)} &= [\kappa S_{x1}, -\kappa C_{x1}, \kappa S_{x2}, -\kappa C_{x2}, 0_{1,4}]^T, C^{(3)} = 0_{8,1} \\
C^{(4)} &= [0_{1,4}, \kappa S_{y1}, -\kappa C_{y1}, \kappa S_{y2}, -\kappa C_{y2}]^T, C^{(5)} = 0_{8,1}, C^{(6)} = [C^{(61)}, C^{(62)}]^T \\
C^{(61)} &= \kappa l_x C_{\theta yaw} [S_{x1}, -C_{x1}, -S_{x2}, C_{x2}], C^{(62)} = \kappa l_y C_{\theta yaw} [S_{y1}, -C_{y1}, -S_{y2}, C_{y2}] \\
B_{elec2} &= -(R + k_P + k_I) \text{diag}[1, 1, 1, 1, 1, 1, 1, 1] \\
A_{NOB} &= [A_{NOB1}, A_{NOB2}] \\
A_{NOB1} &= [C^{(7)}, C^{(8)}, C^{(9)}, C^{(10)}, C^{(11)}, C^{(12)}], C^{(7)} = [C^{(71)}, C^{(72)}]^T \\
C^{(71)} &= [-l_{\theta x}, -l_{\omega x}, 0_{1,4}], C^{(72)} = [-l_{x1a}, -l_{x1b}, -l_{x2a}, -l_{x2b}, 0_{1,4}], C^{(8)} = [C^{(81)}, C^{(82)}]^T \\
C^{(81)} &= [1, -\frac{B_x}{M}, 0_{1,4}], C^{(82)} = \frac{\kappa}{L} [S_{x1}, -C_{x1}, S_{x2}, -C_{x2}, 0_{1,4}], C^{(9)} = [C^{(91)}, C^{(92)}]^T \\
C^{(91)} &= [0_{1,2}, -l_{\theta y}, -l_{\omega y}, 0_{1,2}], C^{(92)} = \frac{\kappa}{L} [0_{1,4}, -l_{y1a}, -l_{y1b}, -l_{y2a}, -l_{y2b}] \\
C^{(10)} &= [C^{(10a)}, C^{(10b)}]^T, C^{(10a)} = [0_{1,2}, 1, -\frac{B_y}{M}, 0_{1,2}] \\
C^{(10b)} &= \frac{\kappa}{L} [0_{1,4}, S_{y1}, -C_{y1}, S_{y2}, -C_{y2}], C^{(11)} = [C^{(11a)}, C^{(11b)}]^T \\
C^{(11a)} &= [0_{1,4}, -l_{\theta yaw}, -l_{\omega yaw}], C^{(11b)} = 0_{1,8}, C^{(12)} = [C^{(12a)}, C^{(12b)}]^T \\
C^{(12a)} &= [0_{1,4}, 1, -\frac{B_{yaw}}{J}], C^{(12b)} = [C^{(12b1)}, C^{(12b2)}] \\
C^{(12b1)} &= \frac{\kappa l_x C_{\theta yaw}}{L} [S_{x1}, -C_{x1}, -S_{x2}, C_{x2}], C^{(12b2)} = \frac{\kappa l_y C_{\theta yaw}}{L} [S_{y1}, -C_{y1}, -S_{y2}, C_{y2}] \\
A_{NOB2} &= [R^{(1)}, R^{(2)}]^T, R^{(1)} = [R^{(11)}, R^{(12)}, R^{(13)}, R^{(14)}, R^{(15)}, R^{(16)}]^T, R^{(11)} = 0_{1,8} \\
R^{(12)} &= [-\frac{\kappa}{M} S_{x1}, \frac{\kappa}{M} C_{x1}, -\frac{\kappa}{M} S_{x2}, \frac{\kappa}{M} C_{x2}, 0_{1,4}], R^{(13)} = 0_{1,8} \\
R^{(14)} &= [0_{1,4}, -\frac{\kappa}{M} S_{y1}, \frac{\kappa}{M} C_{y1}, -\frac{\kappa}{M} S_{y2}, \frac{\kappa}{M} C_{y2}], R^{(15)} = 0_{1,8}, R^{(16)} = [R^{(16a)}, R^{(16b)}] \\
R^{(16a)} &= \frac{\kappa l_x}{J} [-S_{x1}, C_{x1}, S_{x2}, -C_{x2}], R^{(16b)} = \frac{\kappa l_y}{J} [-S_{y1}, C_{y1}, S_{y2}, -C_{y2}] \\
R^{(2)} &= -\frac{R}{L} \text{diag}[1, 1, 1, 1, 1, 1, 1, 1] \\
S_{x1} &= \sin(\gamma \theta_{x1}), C_{x1} = \cos(\gamma \theta_{x1}), S_{x2} = \sin(\gamma \theta_{x2}), C_{x2} = \cos(\gamma \theta_{x2}) \\
S_{y1} &= \sin(\gamma \theta_{y1}), C_{y1} = \cos(\gamma \theta_{y1}), S_{y2} = \sin(\gamma \theta_{y2}), C_{y2} = \cos(\gamma \theta_{y2}) \\
C_{\theta yaw} &= \cos(\theta_{yaw})
\end{aligned}$$

References

1. Krishnamurthy, P.; Khorrami, F. Robust adaptive control of Sawyer motors without current measurements. *IEEE/ASME Trans. Mechatron.* **2004**, *9*, 689–696. [\[CrossRef\]](#)
2. Jones, D.W. Control of stepping motors. In *Handbook of Small Electric Motors*; Yeadon, W.H., Yeadon, A.W., Eds.; McGraw-Hill: New York, NY, USA, 2001.
3. Kim, W.H.; Shin, D.H.; Chung, C.C. Microstepping with nonlinear torque modulation for permanent magnet stepper motors. *IEEE Trans. Control Syst. Technol.* **2013**, *21*, 1971–1979.
4. Lee, Y.W.; Shin, D.H.; Kim, W.H.; Chung, C.C. Proximate in-phase current estimator to reduce torque ripple in permanent-magnet stepping motor. *IEEE Trans. Ind. Electron.* **2016**, *63*, 1707–1716. [\[CrossRef\]](#)
5. Gaan, D.R.; Kumar, M.; Sudhakar, S. Real-time precise position tracking with stepper motor using frequency modulation based microstepping. *IEEE Trans. Ind. Appl.* **2018**, *54*, 693–701. [\[CrossRef\]](#)
6. Pan, J.; Cheung, N.C.; Yang, J. High-precision position control of a novel planar switched reluctance motor. *IEEE Trans. Ind. Electron.* **2005**, *52*, 1644–1652. [\[CrossRef\]](#)
7. Kim, W.H.; Shin, D.H.; Lee, Y.W. Nonlinear position control using only position feedback under position errors and yaw constraints for air bearing planar motors. *Mathematics* **2020**, *8*, 1354. [\[CrossRef\]](#)
8. Su, K.H.; Byeon, K.K.; Kim, W.H.; Lee, Y.W. LPV H_∞ control with an augmented nonlinear observer for sawyer motors. *Mathematics* **2022**, *10*, 18. [\[CrossRef\]](#)
9. Zhao, J.; Wang, L.; Dong, F.; He, Z.; Song, J. Robust high bandwidth current regulation for permanent magnet synchronous linear motor drivers by using two-degree-of-freedom controller and thrust ripple observer. *IEEE Trans. Ind. Electron.* **2020**, *67*, 1804–1812. [\[CrossRef\]](#)
10. Qu, L.; Qiao, W.; Qu, L. Active-disturbance-rejection-based sliding-mode current control for permanent-magnet. *IEEE Trans. Power Electron.* **2021**, *36*, 751–760. [\[CrossRef\]](#)
11. Hou, L.; Ma, J.; Wang, W. Sliding mode predictive current control of permanent magnet synchronous motor with cascaded variable rate sliding mode speed controller. *IEEE Access* **2022**, *10*, 33992–34002. [\[CrossRef\]](#)
12. Le, K.M.; Van, H.H.; Jeon, J.W. An advanced closed-loop control to improve the performance of hybrid stepper motors. *IEEE Trans. Power Electron.* **2017**, *32*, 7244–7255. [\[CrossRef\]](#)
13. Cui, Z.; Dai, J.; Sun, J.; Li, D.; Wang, L.; Wang, K. Hybrid methods using neural network and kalman filter for the state of charge estimation of lithium-ion battery. *Math. Probl. Eng.* **2022**, *2022*, 9616124. [\[CrossRef\]](#)
14. Li, D.; Li, S.; Zhang, S.; Sun, J.; Wang, L.; Wang, K. Aging state prediction for supercapacitors based on heuristic kalman filter optimization extreme learning machine. *Energy* **2022**, *250*, 123773. [\[CrossRef\]](#)
15. Hua, Y.; Wang, N.; Zhao, K. Simultaneous unknown input and state estimation for the linear system with a rank-deficient distribution matrix. *Math. Probl. Eng.* **2021**, *2021*, 6693690. [\[CrossRef\]](#)
16. Arora, S.; Balsara, P.; Bhatia, D. Input—Output linearization of a boost converter with mixed load (Constant voltage load and constant power load). *IEEE Trans. Power Electron.* **2019**, *34*, 815–825. [\[CrossRef\]](#)
17. Mehra, M.; Babaie, M.; Sharifzadeh, M.; Haddad, K.A. An input—Output feedback linearization control method synthesized by artificial neural network. *IEEE Trans. Ind. Appl.* **2021**, *57*, 3131–3142. [\[CrossRef\]](#)
18. Accetta, A.; Alonge, F.; Cirrincione, M.; D’Ippolito, F.; Pucci, M.; Rabbeni, R.; Sferlazza, A. Robust control for high performance induction motor drives based on partial state-feedback linearization. *IEEE Trans. Ind. Appl.* **2019**, *55*, 490–503. [\[CrossRef\]](#)
19. Li, H.; Wang, Z.; Xu, Z.; Wang, X.; Hu, Y. Feedback linearization based direct torque control for IPMSMs. *IEEE Trans. Power Electron.* **2021**, *36*, 3135–3148. [\[CrossRef\]](#)
20. Khalil, H.K. *Nonlinear Systems*, 3rd ed.; Prentice-Hall: Englewood Cliffs, NJ, USA, 2002.