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BOUNDS FOR GENERALIZED FRACTIONAL INTEGRAL OPERATORS OF GENERALIZED STRONGLY CONVEX FUNCTIONS VIA MITTAG-LEFFLER FUNCTIONS

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Abstract

In this paper, we investigate bounds of the fractional integral operators containing the unified Mittag-Leffler function in their kernels. These bounds are given by applying a generalized class of functions called strongly exponentially $(\alpha, h-m)$-convex functions, which give refinements of various types of convexities. Utilized integral operators (2) and (3) unify a lot of well known fractional integral operators. Therefore, the results of this paper contain refinements as well as generalizations of many well-known fractional integral inequalities, which have been published in different articles in recent past.

Keywords: Strongly Exponentially $(\alpha, h-m)$-Convex Function; Unified Mittag-Leffler Function; Generalized Fractional Integral Operators.

1. INTRODUCTION

Like a function in the subject of real analysis, an operator in the subject of functional analysis is equally important. In fact, an operator is a function with domain and range vector spaces. For instance, differentiation and definite integral are basic examples of operators, and the operators defined with the help of these operators are known as differential operators, and integral operators, respectively. These operators are linear, and operators having property of linearity preserve the continuity in usual. Boundedness of operators is another important property which leads to Banach algebra. Banach algebras with some additional structure play a very vital role in quantum mechanics. There are various types of operators that exist in literature which are used in diverse fields of pure and applied nature, see Refs. 1 and 2.

Fractional integral operators have gained importance in several domains of science and engineering over the past four decades. The theory of fractional calculus based on these operators is recognized to address different complex problems occur in mathematics, physics, fluid mechanics, optics, and signals processing. Fractional integral operators are commonly utilized for generalizing classical and new integral inequalities. For instance, well-known inequalities of Chebyshev type,[3,4] Ostrowski type,[5–8] Minkowski type,[9,10] Opial type,[11] Grüss type,[12] Hadamard type[13–16] and Fejér type,[17] for different kinds of fractional integrals were published in recent years.

The well-known Mittag-Leffler function plays an important role for solving fractional differential equations, fractional dynamical systems and modeling fractional-order viscoelastic materials, see Refs. 18–20. It is also used in generalizing the integral as well as fractional integral operators. For a detail study, we refer the readers to Ref. 21. Inequalities for fractional integral operators containing Mittag-Leffler functions have been studied in Refs. 22–24.

In this paper, we aim to establish bounds of the fractional integral operators containing the unified Mittag-Leffler function given in Ref. 25. To attain the desired fractional inequalities, definition of strongly exponentially $(\alpha, h-m)$-convex functions is applied. The proving theorems are generalizations and refinements of several recently published results. Next, we define the unified Mittag-Leffler function.

Definition 1 (Ref. 25). With conditions as in Definition given in Ref. 25, we have the unified

Mittag-Leffler function as follows:

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b},\underline{c},\varrho) = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B_{\varrho}(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl}z^l}{\prod_{i=1}^{n}B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)},$$

where $\Re(x)$ denotes the real part of x, B denotes the beta function and B_ϱ is the extension of beta function defined as follows:

$$B_{\varrho}(\varsigma,y) = \int_0^1 \zeta^{\varsigma-1}(1-\zeta)^{y-1}e^{-\frac{\varrho}{\zeta(1-\zeta)}}d\zeta.$$

In the following, we define generalized fractional integral operators containing the unified Mittag-Leffler function.

Definition 2 (Ref. 25). With same conditions as in Ref. 25, we have unified operators as follows:

$$(\Upsilon^{\tau,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{a},\underline{b},\underline{c},\varrho) = \int_{\xi_1}^{\varsigma}(\varsigma-\zeta)^{\alpha-1}M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\varsigma-\zeta)^{\mu};\underline{a},\underline{b},\underline{c},\varrho)\Phi(\zeta)d\zeta,$$

$$(\Upsilon^{\tau,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{a},\underline{b},\underline{c},\varrho) = \int_{\varsigma}^{\xi_2}(\zeta-\varsigma)^{\alpha-1}M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\varsigma-\zeta)^{\mu};\underline{a},\underline{b},\underline{c},\varrho)\Phi(\zeta)d\zeta.$$

By setting $a_i = l$, $\varrho = 0$ and $\Re(\rho) > 0$ in Definition 2, we get the fractional integral operators associated with generalized Q function as follows (see, Ref. 26):

$$({}_{Q}\Upsilon^{\tau,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{b},\underline{c}) = \int_{\xi_1}^{\varsigma}(\varsigma-\zeta)^{\alpha-1}Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\varsigma-\zeta)^{\mu};\underline{b},\underline{c})\Phi(\zeta)d\zeta,$$

$$({}_{Q}\Upsilon^{\tau,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{b},\underline{c}) = \int_{\varsigma}^{b}(\zeta-\varsigma)^{\alpha-1}Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\zeta-\varsigma)^{\mu};\underline{b},\underline{c})\Phi(\zeta)d\zeta,$$

where

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{b},\underline{c}) = \sum_{l=0}^{\infty}\frac{\prod_{i=1}^{n}B(c_i,l)(\lambda)_{\rho l}(\theta)_{kl}z^l}{\prod_{i=1}^{n}B(b_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}. \tag{1}$$

In Ref. 27, there is given an extended definition, here we leave the convergence conditions.

Definition 3. Let $\Delta \in L_1[\xi_1,\xi_2]$, $0<\xi_1<\xi_2<\infty$ be a positive function and let $\Psi:[\xi_1,\xi_2]\to\mathbb{R}$ be a differentiable and strictly increasing function. Also, let $\frac{\Delta(\varsigma)}{\varsigma}$ be an increasing function on $[\xi_1,\infty)$ and $\varsigma\in[\xi_1,\xi_2]$. Then the unified integral operator is defined by the following integrals:

$$({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\varrho) = \int_{\xi_1}^{\varsigma}\Lambda^{\zeta}_{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\zeta)d(\Psi(\zeta)), \tag{2}$$

$$({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\varrho) = \int_{\varsigma}^{\xi_2}\Lambda^{\varsigma}_{\zeta}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\zeta)d(\Psi(\zeta)), \tag{3}$$

where

$$\Lambda^{\zeta}_{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\Delta) = \frac{\Delta(\Psi(\varsigma)-\Psi(\zeta))}{\Psi(\varsigma)-\Psi(\zeta)}M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\Psi(\varsigma)-\Psi(\zeta))^{\mu};\underline{a},\underline{b},\underline{c},\varrho).$$

Let us put $a_i = l$, $\varrho = 0$ and consider $\Re(\rho) > 0$ in (2) and (3), we obtain integral operators containing Q function, see, Ref. 26:

$$({}^{\Psi}_{Q}\Upsilon^{\Delta,\tau,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{b},\underline{c}) = \int_{\xi_1}^{\varsigma}\Phi(\zeta)\Lambda^{\zeta}_{\varsigma}(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\Delta)d(\Psi(\zeta)),$$

$$({}^{\Psi}_{Q}\Upsilon^{\Delta,\tau,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\underline{b},\underline{c}) = \int_{\varsigma}^{\xi_2}\Phi(\zeta)\Lambda^{\zeta}_{\varsigma}(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Psi;\Delta)d(\Psi(\zeta)),$$

where

$$\Lambda^{\zeta}_{\varsigma}(Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta) = \frac{\Delta(\Psi(\varsigma)-\Psi(\zeta))}{\Psi(\varsigma)-\Psi(\zeta)}Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu} \times(\tau(\Psi(\varsigma)-\Psi(\zeta))^{\mu};\underline{b},\underline{c},\varrho).$$

It is easy to note that if Ψ and $\frac{\Delta}{\varsigma}$ are increasing, then for $u < \zeta < v, u, v \in [\xi_1, \xi_2]$, the kernel $({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varsigma;\varrho)$ satisfies the following inequality:

$$\Lambda^u_\zeta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta) \leq \Lambda^u_v(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta). \quad (4)$$

A function f satisfying the following inequality (5) is called a strongly exponentially $(\alpha, h-m)$-convex function, see details in Ref. 30.

$$\begin{aligned} f(\varsigma\zeta + m(1-\zeta)y) &\leq h(\zeta^\alpha)\frac{f(\varsigma)}{e^{\eta\varsigma}} + mh(1-\zeta^\alpha)\frac{f(y)}{e^{\eta y}} \\ &\quad - \frac{Cm}{e^{\eta(\varsigma+y)}}h(\zeta^\alpha)h(1-\zeta^\alpha)|y-\varsigma|^2. \quad (5)\end{aligned}$$

Motivated by recent developments in fractional calculus operators of derivatives/integrals, and extensions of generalized convexities, this paper explores inequalities for recently defined unified fractional integral operators (2) and (3), and a generalized strongly convex function satisfying (5). Our goal is to unify bounds of integral operators of fractional orders defined in recent past decades for many kinds of strongly convex functions. It is important to mention, the results of this paper reproduce inequalities published in Refs. 28–30. Also, in special cases refinements of bounds of all associated fractional integral operators can be deduced.

In the upcoming section, we give bounds of fractional integral operators (2) and (3) by using strongly exponentially $(\alpha, h-m)$-convexity. Moreover, we provide the Hadamard type inequality by using a symmetry like condition. The outcomes of this paper give generalizations of a number of inequalities that have been published in near past.[29] Lastly, a result for differentiable function Φ such that $|\Phi'|$ is strongly exponentially $(\alpha, h-m)$-convex is obtained.

2. MAIN RESULTS

The first result gives an upper bound of unified fractional integral operators.

Theorem 4. *Let $\Phi, \Delta \in L_1[\xi_1, \xi_2]$ and Φ be strongly exponentially $(\alpha, h-m)$-convex function with $0 \leq \xi_1 < m\xi_2$. Also, let $\Psi : [\xi_1, \xi_2] \to \mathbb{R}$ be differentiable and strictly increasing function with the assumption that $\frac{\Delta(\varsigma)}{\varsigma}$ be an increasing function on $[\xi_1, \xi_2]$. Then for $\varsigma \in [\xi_1, \xi_2]$, we have the upcoming inequality:*

$$\begin{aligned} &({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\varsigma;\varrho) + ({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\varsigma;\varrho) \\ &\leq \Lambda^{\xi_1}_\varsigma(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)(\varsigma-\xi_1) \\ &\quad\times\left(\frac{\Phi(\xi_1)}{e^{\eta\xi_1}}X^{\xi_1}_\varsigma(\sigma^\alpha,h,\Psi') + \frac{m}{e^{\frac{\eta\varsigma}{m}}}\right. \\ &\quad\times\Phi\left(\frac{\varsigma}{m}\right)X^{\xi_1}_\varsigma(1-\sigma^\alpha,h,\Psi') \\ &\quad\left.-\frac{C(\varsigma-\xi_1 m)^2h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right) \\ &\quad+(\xi_2-\varsigma)\Lambda^\varsigma_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi;\Delta)\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X^{\xi_2}_\varsigma\right. \\ &\quad\times(\sigma^\alpha,h,\Psi') + \frac{m}{e^{\frac{\eta\varsigma}{m}}}\Phi\left(\frac{\varsigma}{m}\right)X^{\xi_2}_\varsigma(1-\sigma^\alpha,h,\Psi') \\ &\quad\left.-\frac{C(\xi_2 m-\varsigma)^2h(1)(\Psi(\xi_2)-\Psi(\varsigma))}{m(\xi_2-\varsigma)e^{\eta}(\xi_2+\frac{\varsigma}{m})}\right), \quad (6)\end{aligned}$$

where $X^{\xi_1}_\varsigma(\sigma^\alpha,h,\Psi') = \int_0^1 h(\sigma^\alpha)\Psi'(\varsigma - r(\varsigma-\xi_1))dr$, $X^{\xi_1}_\varsigma(1-\sigma^\alpha,h,\Psi') = \int_0^1 h(1-\sigma^\alpha)\Psi'(\varsigma-r(\varsigma-\xi_1))dr$.

Proof. By using inequality (4), one can have the following inequalities:

$$\begin{aligned} &\Lambda^\zeta_\varsigma(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\Delta)\Psi'(\zeta) \\ &\quad\leq \Lambda^{\xi_1}_\varsigma(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi;\Delta)\Psi'(\zeta), \quad (7)\\ &\Lambda^\varsigma_\zeta(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\zeta) \\ &\quad\leq \Lambda^\varsigma_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\ \Psi;\Delta)\Psi;\Delta)\Psi'(\zeta). \quad (8)\end{aligned}$$

Applying (5) to the function Φ, one can yield:

$$\begin{aligned} \Phi(\zeta) &\leq h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^\alpha\frac{\Phi(\xi_1)}{e^{\eta\xi_1}} + mh\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^\alpha\right) \\ &\quad\times\frac{\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}} - \frac{C(\varsigma-\xi_1 m)^2}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^\alpha \\ &\quad\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^\alpha\right) \quad (9)\end{aligned}$$

and

$$\begin{aligned} \Phi(\zeta) &\leq h\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^\alpha\frac{\Phi(\xi_2)}{e^{\eta\xi_2}} + mh\left(1-\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^\alpha\right) \\ &\quad\times\frac{\Phi\left(\frac{\varsigma}{m}\right)}{e^{\frac{\eta\varsigma}{m}}} - \frac{C(\xi_2 m-\varsigma)^2}{me^{\eta(\xi_2+\frac{\varsigma}{m})}}h\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^\alpha \\ &\quad\times h\left(1-\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^\alpha\right). \quad (10)\end{aligned}$$

The following inequality can be found by integrating the product of inequalities (7) and (9) over $[\xi_1, \varsigma]$:

$$\int_{\xi_1}^{\varsigma} \Lambda_{\varsigma}^{\zeta}(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)\Phi(\zeta)d(\Psi(\zeta))$$

$$\leq \Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)\left(\frac{\Phi(\xi_1)}{e^{\eta\xi_1}}\right.$$

$$\times \int_{\xi_1}^{\varsigma} h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha} d(\Psi(\zeta)) + \frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}$$

$$\times \int_{\xi_1}^{\varsigma} h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right) d(\Psi(\zeta))$$

$$- \frac{C(\varsigma-\xi_1 m)^2}{me^{\eta(\xi_1+\frac{\varsigma}{m})}} \int_{\xi_1}^{\varsigma} h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$

$$\left.\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_2}\right)^{\alpha}\right) d(\Psi(\zeta))\right).$$

Using Definition 3 and setting $\sigma = \frac{\varsigma-\zeta}{\varsigma-\xi_1}$, we obtain the following inequality:

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi)(\varsigma;\varrho)$$

$$\leq \Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)(\varsigma-\xi_1)\left(\frac{\Phi(\xi_1)}{e^{\eta\xi_1}}\right.$$

$$\times \int_0^1 h(\sigma^{\alpha})\Psi(\varsigma-r(\varsigma-\xi_1))dr + \frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}$$

$$\times \int_0^1 h(1-\sigma^{\alpha})\Psi'(\varsigma-r(\varsigma-\xi_1)))dr$$

$$- \frac{C(\varsigma-\xi_1 m)^2}{me^{\eta(\xi_1+\frac{\varsigma}{m})}} \int_0^1 h(\sigma^{\alpha})h(1-\sigma^{\alpha})$$

$$\left.\times \Psi'(\varsigma-r(\varsigma-\xi_1))dr\right).$$

Hence, we get

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi)(\varsigma;\varrho)$$

$$\leq \Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)\left(\frac{\Phi(\xi_1)}{e^{\eta\xi_1}}X_{\varsigma}^{\xi_1}(\sigma^{\alpha},h,\Psi')\right.$$

$$+ \frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_1}(\sigma^{\alpha},h,\Psi')$$

$$- \frac{C(\varsigma-\xi_1 m)^2 h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}. \quad (11)$$

Similarly, from (8) and (10) one can find the following inequality:

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi)(\varsigma;\varrho)$$

$$\leq \Lambda_{\xi_2}^{\varsigma}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Phi;\Delta)(\xi_2-\varsigma)\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}\int_0^1 h(\sigma^{\alpha})\right.$$

$$\times \Psi'(\varsigma-r(\xi_2-\varsigma))dr + \frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}$$

$$\times \int_0^1 h(1-\sigma^{\alpha})\Psi'(\varsigma-r(\xi_2-\varsigma))dr$$

$$- \frac{C(m\xi_2-\varsigma)^2}{me^{\eta(\xi_2+\frac{\varsigma}{m})}}\int_0^1 h(\sigma^{\alpha})h(1-\sigma^{\alpha})$$

$$\left.\times \Psi'(\varsigma-r(\xi_1-\varsigma))dr\right).$$

Hence, we get

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi)(\varsigma;\varrho)$$

$$\leq \Lambda_{\xi_2}^{\varsigma}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Phi;\Delta)(\xi_2-\varsigma)$$

$$\times \left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X_{\varsigma}^{\xi_2}(\sigma^{\alpha},h,\Psi')\right.$$

$$+ \frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_2}(1-\sigma^{\alpha},h,\Psi')$$

$$\left.- \frac{C(\varsigma-\xi_1 m)^2 h(1)(\Psi(\xi_2)-\Psi(\varsigma))}{m(\xi_2-\varsigma)e^{\eta(\xi_2+\frac{\varsigma}{m})}}\right). \quad (12)$$

Inequality (6) can be obtained by adding the inequalities (11) and (12). □

The following remark provides consequences of the above theorem.

Remark 5. In (6), the following specific settings: $n = 1$, $b_1 = \lambda + lk$, $a_1 = \theta - \lambda$, $c_1 = \lambda$, $\varrho = \nu = 0$, provide Theorem 7 of Ref. 30, while the settings $\lambda = \eta = 0$, $h(\varsigma) = \varsigma$ provide Theorem 4 of Ref. 28.

To obtain upper and lower bounds of unified fractional integral operators in the form of Hadamard inequality, the following lemma is needful.

Lemma 6 (Ref. 30). *Let $\Phi : [\xi_1, \xi_2] \to \mathbb{R}$ be a strongly exponentially $(\alpha, h-m)$-convex function with modulus $C \geq 0$, $m \in (0, 1]$, $0 \leq \xi_1 < m\xi_2$. If $\frac{\Phi(\varsigma)}{e^{\varsigma}} = \frac{\Phi(\frac{\xi_1+m\xi_2-\varsigma}{m})}{e^{\frac{\xi_1+m\xi_2-\varsigma}{m}}}$, then we have:*

$$\Phi\left(\frac{\xi_1+m\xi_2}{2}\right) \leq \left(h\left(\frac{1}{2^{\alpha}}\right) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\frac{\Phi(\varsigma)}{e^{\eta\varsigma}}$$

$$- \frac{C}{m}h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)$$

$$\times \frac{(\xi_1-\varsigma+m\xi_2-m\varsigma)^2}{e^{\frac{\eta(\xi_1+m\xi_2+\varsigma(m-1))}{m}}}. \quad (13)$$

The Hadamard type inequality is stated and proved in the upcoming theorem.

Theorem 7. *Let* $\frac{\Phi(\varsigma)}{e^{\varsigma}} = \frac{\Phi(\frac{\xi_1+m\xi_2-\varsigma}{m})}{e^{\frac{\xi_1+m\xi_2-\varsigma}{m}}}$ *along assumptions of Theorem 4. Then we have*

$$\frac{A(\eta)}{h(\frac{1}{2^\alpha})+mh(\frac{2^\alpha-1}{2^\alpha})}\Bigg[\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)$$

$$\times\left(\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}1\right)(\xi_1;\varrho)+\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}1\right)\right.$$

$$\left.\times(\xi_2;\varrho)\right)+\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)$$

$$\left(\frac{\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\varsigma+m\xi_2-m\varsigma)^2\right)(\xi_1;\varrho)}{e^{\eta(\frac{2m\xi_2+\xi_1-\xi_2}{m})}}\right.$$

$$\left.+\frac{\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\varsigma+m\xi_2-m\varsigma)^2\right)\times(\xi_1;\varrho)}{e^{\eta(\xi_1+\xi_2)}}\right)\Bigg]$$

$$\leq\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(\xi_1;\varrho)+\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)$$

$$\times(\xi_2;\varrho)$$

$$\leq(\xi_2-\xi_1)\left(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\right.$$

$$\left.+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\right)\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X^{\xi_1}_{\xi_2}(\sigma^\alpha,h;\Psi')\right.$$

$$+\frac{m\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}X^{\xi_1}_{\xi_2}(1-\sigma^\alpha,h;\Psi')$$

$$\left.-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)e^{\eta(\xi_2+\frac{\xi_1}{m})}}\right),\qquad(14)$$

where $A(\eta)=e^{\eta\xi_1}$ *for* $\eta\geq 0$ *and* $A(\eta)=e^{\eta\xi_2}$ *for* $\eta<0$.

Proof. By using (4), one can have the following inequalities:

$$\Lambda^{\xi_1}_{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\varsigma)$$

$$\leq\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\varsigma),\qquad(15)$$

$$\Lambda^{\varsigma}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\varsigma)$$

$$\leq\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\varsigma).\qquad(16)$$

Applying (5) to Φ, we have

$$\Phi(\varsigma)\leq h\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}$$

$$+mh\left(1-\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha\right)\frac{\Phi(\frac{\varsigma}{m})}{e^{\frac{\eta\varsigma}{m}}}$$

$$-\frac{C(\xi_1-m\xi_2)^2}{me^{\eta(\xi_2+\frac{\xi_1}{m})}}h\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha h$$

$$\times\left(1-\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha\right).\qquad(17)$$

The following inequality can be found by integrating the product of inequalities (15) and (17) over $[\xi_1,\xi_2]$:

$$\int_{\xi_1}^{\xi_2}\Lambda^{\xi_1}_{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Phi(\varsigma)d(\Psi(\varsigma))$$

$$\leq\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$$

$$\times\left(\frac{m\Phi(\frac{\xi_1}{m})}{e^{\frac{\eta\xi_1}{m}}}\int_{\xi_1}^{\xi_2}h\left(1-\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha\right)d(\Psi(\varsigma))\right.$$

$$+\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}\int_{\xi_1}^{\xi_2}h\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha d(\Psi(\varsigma))$$

$$-\frac{C(\xi_1-m\xi_2)^2}{me^{\eta(\xi_2+\frac{\xi_1}{m})}}\int_{\xi_1}^{\xi_2}h\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha$$

$$\left.\times h\left(1-\left(\frac{\varsigma-\xi_1}{\xi_2-\xi_1}\right)^\alpha\right)d(\Psi(\varsigma))\right).$$

By using Definition 3 and setting $r=\frac{\varsigma-\xi_1}{\xi_2-\xi_1}$, the following inequality is obtained:

$$\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(\xi_1;\varrho)$$

$$\leq\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)(\xi_2-\xi_1)\left(\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X^{\xi_1}_{\xi_2}\right.\right.$$

$$\left.\times(\sigma^\alpha,h;\Psi')+\frac{m}{e^{\frac{\eta\varsigma}{m}}}\Phi\left(\frac{\varsigma}{m}\right)X^{\xi_1}_{\xi_2}(1-\sigma^\alpha,h;\Psi')\right.$$

$$\left.-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)e^{\eta(\xi_2+\frac{\xi_1}{m})}}\right).\qquad(18)$$

Similarly, from (17) and (16) one can find the following inequality:

$$\left({}^{\Delta}_{\Psi}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi\right)(\xi_2;\varrho)$$

$$\leq\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)(\xi_2-\xi_1)$$

$$\times\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X_{\xi_2}^{\xi_1}(\sigma^\alpha,h;\Psi')+\frac{m}{e^{\frac{\eta\varsigma}{m}}}\Phi\left(\frac{\varsigma}{m}\right)\right.$$
$$\times X_{\xi_2}^{\xi_1}(1-\sigma^\alpha,h;\Psi')$$
$$\left.-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)}\right). \quad (19)$$

The following inequality can be obtained by adding (18) and (19):

$$(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\xi_1;\varrho)+(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\xi_2;\varrho)$$
$$\le(\xi_2-\xi_1)(\Lambda_{\xi_2}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$$
$$+\Lambda_{\xi_2}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta))\left(\frac{\Phi(\xi_2)}{e^{\eta\xi_2}}X_{\xi_2}^{\xi_1}(\sigma^\alpha,h;\Psi')\right.$$
$$+\frac{m}{e^{\frac{\eta\varsigma}{m}}}\Phi\left(\frac{\varsigma}{m}\right)X_{\xi_2}^{\xi_1}(1-\sigma^\alpha,h;\Psi')$$
$$\left.-\frac{C(\xi_1-m\xi_2)^2h(1)(\Psi(\xi_2)-\Psi(\xi_1))}{m(\xi_2-\xi_1)e^{\eta(\xi_2+\frac{\xi_1}{m})}}\right). \quad (20)$$

Multiplying both sides of (13) by $\Lambda_{\varsigma}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)\Psi'(\varsigma)$ and integrating over $[\xi_1,\xi_2]$, we have

$$\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\int_{\xi_1}^{\xi_2}\Lambda_{\varsigma}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)d(\Psi(\varsigma))$$
$$\le h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{\xi_1}^{\xi_2}\Lambda_{\varsigma}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$$
$$\times\frac{\Phi(\varsigma)}{e^{\eta\varsigma}}d(\Psi(\varsigma))-\frac{C}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)$$
$$\times\int_{\xi_1}^{\xi_2}\Lambda_{\varsigma}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$$
$$\times\frac{(\xi_1-\varsigma+m\xi_2-m\varsigma)^2}{e^{\frac{\eta(\xi_1+m\xi_2+\varsigma(m-1))}{m}}}d(\Psi(\varsigma)).$$

Definition 3 leads to the following inequality:

$$\frac{e^{\eta\xi_1}}{h(\frac{1}{2^\alpha})+mh(\frac{2^\alpha-1}{2^\alpha})}\left(\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\right.$$
$$\times(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}1)(\xi_1;\varrho)+\frac{C}{me^{\eta(\xi_1+\xi_2)}}h\left(\frac{1}{2^\alpha}\right)$$
$$\times h\left(\frac{2^\alpha-1}{2^\alpha}\right)(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\varsigma$$
$$\left.+m\xi_2-m\varsigma)^2)(\xi_1;\varrho)\right)$$
$$\le(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\xi_1;\varrho). \quad (21)$$

Now, multiplying (13) with $\Lambda_{\xi_2}^{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)$ $\Psi'(\varsigma)$ and integrating over $[\xi_1,\xi_2]$, we have

$$\frac{e^{\eta\xi_2}}{h(\frac{1}{2^\alpha})+mh(\frac{2^\alpha-1}{2^\alpha})}\left(\Phi\left(\frac{\xi_1+m\xi_2}{2}\right)\right.$$
$$\times(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}1)(\xi_2;\varrho)+\frac{Ch(\frac{1}{2^\alpha})h(\frac{2^\alpha-1}{2^\alpha})}{me^{\eta(\frac{2m\xi_2+\xi_1-\xi_2}{m})}}$$
$$\left.\times(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\xi_1-\varsigma+m\xi_2-m\varsigma)^2)(\xi_2;\varrho)\right)$$
$$\le(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}\Phi)(\xi_1;\varrho). \quad (22)$$

From (20), (21) and (22), inequality (14) can be obtained. □

Consequences of above theorem are given in the following remark.

Remark 8. In (14), the following specific settings: $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\varrho=\nu=0$, provide Theorem 8 of Ref. 30, while the settings $\lambda=\eta=0$, $h(\varsigma)=\varsigma$, provide Theorem 2 of Ref. 28.

In the next result, a modulus inequality is presented which provides boundedness of fractional integral operator of convolution of two functions.

Theorem 9. *Let $\Phi:[\xi_1,\xi_2]\to\mathbb{R}$ be a differentiable function. If $|\Phi'|$ is strongly exponentially $(\alpha,h-m)$-convex and let $\Psi:[\xi_1,\xi_2]\to\mathbb{R}$ be differentiable and strictly increasing function with $0\le\xi_1<m\xi_2$, also let $\frac{\Delta}{\varsigma}$ be an increasing function on $[\xi_1,\xi_2]$. Then $\forall\ \varsigma\in[\xi_1,\xi_2]$, we have the following inequality:*

$$|(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi*\Psi)(\varsigma;\varrho)$$
$$+(_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi*\Psi)(\varsigma;\varrho)|$$
$$\le\Lambda_{\varsigma}^{\xi_1}(M^{\tau,\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\ \Psi;\Delta)(\varsigma-\xi_1)$$
$$\times\left(\left(\frac{m}{e^{\frac{\eta\varsigma}{m}}}\left|\Phi'\left(\frac{\varsigma}{m}\right)\right|X_{\varsigma}^{\xi_1}(1-\sigma^\alpha,h,\Psi')\right.\right.$$
$$\left.-\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}X_{\varsigma}^{\xi_1}(\sigma^\alpha,h,\Psi')\right)$$
$$\left.-\frac{C(\varsigma-\xi_1m)^2h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right)$$
$$+\Lambda_{\xi_2}^{\varsigma}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\Delta)(\xi_2-\varsigma)$$

$$\times\left(\left(\left(m\frac{\left|\Phi'\left(\frac{\varsigma}{m}\right)\right|}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_2}(1-\sigma^{\alpha},h,\Psi')\right.\right.\right.$$

$$\left.-\frac{|\Phi'(\xi_2)|}{e^{\eta\xi_2}}X_{\varsigma}^{\xi_2}(\sigma^{\alpha},h,\Psi')\right)$$

$$\left.-\frac{C(\xi_2 m-\varsigma)^2h(1)(\Psi(\xi_2)-\Psi(\varsigma))}{m(\xi_2-\varsigma)e^{\eta(\xi_2+\frac{\varsigma}{m})}}\right),\qquad(23)$$

where

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi*\Psi)(\varsigma;\varrho)$$
$$:=\int_{\xi_1}^{\varsigma}\Lambda_{\varsigma}^{\zeta}(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\Delta)\Phi'(\zeta)d(\Psi(\zeta)),$$
$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi*\Psi)(\varsigma;\varrho)$$
$$:=\int_{\varsigma}^{\xi_2}\Lambda_{\zeta}^{\varsigma}(M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)\Phi'(\zeta)d(\Psi(\zeta)).$$

Proof. The function $|\Phi'|$ must satisfy the inequality (5) that is

$$|\Phi'(\zeta)|\le h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}$$
$$+mh\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}$$
$$-\frac{C(\varsigma-\xi_1 m)}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$
$$\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right).$$

This is equivalent to

$$-\left(h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}+mh\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)\right.$$
$$\times\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}-\frac{C(\varsigma-\xi_1 m)}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$
$$\left.\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)\right)\le\Phi'(\zeta)$$
$$\le h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}+mh\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)$$
$$\times\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}-\frac{C(\varsigma-\xi_1 m)}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$
$$\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right).\qquad(24)$$

By considering the second inequality of (24), we have

$$\Phi'(\zeta)\le h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}$$
$$+mh\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)$$
$$\times\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}-\frac{C(\varsigma-\xi_1 m)}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$
$$\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right).\qquad(25)$$

The following inequality can be found by integrating the product of inequalities (7) and (25) over $[\xi_1,\varsigma]$:

$$\int_{\xi_1}^{\varsigma}\Lambda_{\varsigma}^{\zeta}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)d(\Psi(\zeta))$$
$$\le\Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)$$
$$\times\left(|\Phi'(\xi_1)|\int_{\xi_1}^{\varsigma}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}d(\Psi(\zeta))\right.$$
$$+m\left|\Phi\left(\frac{\varsigma}{m}\right)\right|\int_{\xi_1}^{\varsigma}h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)d(\Psi(\zeta))$$
$$-\frac{C(\varsigma-\xi_1 m)}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}\int_{\xi_1}^{\varsigma}h\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}$$
$$\left.\times h\left(1-\left(\frac{\varsigma-\zeta}{\varsigma-\xi_1}\right)^{\alpha}\right)d(\Psi(\zeta))\right).$$

Hence, we get

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}\Phi*\Psi)(\varsigma;\varrho)$$
$$\le\Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\ \Psi;\Delta)(\varsigma-\xi_1)$$
$$\times\left(\left(m\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_1}(1-\sigma^{\alpha},h,\Psi')\right.\right.$$
$$\left.-\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}X_{\varsigma}^{\xi_1}(\sigma^{\alpha},h,\Psi')\right)$$
$$\left.-\frac{C(\varsigma-\xi_1 m)^2h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right).\qquad(26)$$

Similarly, using the other inequality of (24), one can obtain the following inequality:

$$({}_{\Psi}^{\Delta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}^{\tau,\lambda,\rho,\theta,k,n}(\Phi*\Psi))(\varsigma;\varrho)$$
$$\ge-\Lambda_{\varsigma}^{\xi_1}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n}\Psi;\Delta)(\varsigma-\xi_1)$$

$$\times\left(\left(m\frac{\left|\Phi'\left(\frac{\varsigma}{m}\right)\right|}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_1}(1-\sigma^{\alpha},h,\Psi')\right.\right.$$
$$\left.-\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}X_{\varsigma}^{\xi_1}(\sigma^{\alpha},h,\Psi')\right)$$
$$\left.-\frac{C(\varsigma-\xi_1 m)^2h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right). \quad (27)$$

Inequalities (26), (27) lead to the upcoming inequality:

$$|({}_{\Psi}^{\Delta}\Upsilon^{\tau,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\mu,\nu}(\Phi*\Psi))(\varsigma;\varrho)|$$
$$\leq\Lambda_{\varsigma}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}\Psi;\Delta)(\varsigma-\xi_1)$$
$$\times\left(\left(m\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_1}(1-\sigma^{\alpha},h,\Psi')\right.\right.$$
$$\left.-\frac{|\Phi'(\xi_1)|}{e^{\eta\xi_1}}X_{\varsigma}^{\xi_1}(\sigma^{\alpha},h,\Psi')\right)$$
$$\left.-\frac{C(\varsigma-\xi_1 m)^2h(1)(\Psi(\varsigma)-\Psi(\xi_1))}{m(\varsigma-\xi_1)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right). \quad (28)$$

Now, by applying (5) to the function $|\Phi'|$, we have

$$|\Phi'(\zeta)|\leq h\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^{\alpha}\frac{|\Phi'(\xi_2)|}{e^{\eta\xi_2}}$$
$$+mh\left(1-\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^{\alpha}\right)\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}$$
$$-\frac{C(\varsigma-\xi_1 m)^2}{me^{\eta(\xi_1+\frac{\varsigma}{m})}}h\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^{\alpha}$$
$$\times h\left(1-\left(\frac{\zeta-\varsigma}{\xi_2-\varsigma}\right)^{\alpha}\right). \quad (29)$$

Similarly, from (8) and (29) one can find the following inequality:

$$|({}_{\Psi}^{\Delta}\Upsilon^{\tau,\gamma,\delta,k,c}_{\xi_2^-,\alpha,\beta,\gamma,\mu,\nu}(\Phi*\Psi))(\varsigma;\varrho)|$$
$$\leq\Lambda^{\varsigma}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\Delta)(\xi_2-\varsigma)$$
$$\times\left(\left(m\frac{|\Phi'(\frac{\varsigma}{m})|}{e^{\frac{\eta\varsigma}{m}}}X_{\varsigma}^{\xi_2}(1-\sigma^{\alpha},h,\Psi')\right.\right.$$
$$\left.-\frac{|\Phi'(\xi_2)|}{e^{\eta\xi_2}}X_{\varsigma}^{\xi_2}(\sigma^{\alpha},h,\Psi')\right)$$
$$\left.-\frac{C(\xi_2 m-\varsigma)^2h(1)(\Psi(\xi_2)-\Psi(\varsigma))}{m(\xi_2-\varsigma)e^{\eta(\xi_1+\frac{\varsigma}{m})}}\right). \quad (30)$$

Inequality (23) is obtained by summing (28) and (30). □

Remark 10. In (23), the following specific settings: $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\varrho=\nu=0$, provide Theorem 9 of Ref. 30, while the settings $\lambda=\eta=0$, $h(\varsigma)=\varsigma$ provide Theorem 3 of Ref. 28.

3. CONCLUSION

The bounds of fractional integral operators containing the unified Mittag-Leffler function were investigated via strongly exponentially $(\alpha,h-m)$-convex functions. The established results provided generalizations as well as refinements of many integral inequalities that had been published in recent past. Moreover, the inequalities for strongly exponentially (α,m)-convex, strongly exponentially (s,m)-convex, strongly exponentially $(h-m)$-convex and strongly convex functions could be deduced after setting specific substitutions. The methods of providing bounds of fractional integral operators can be applied for new kinds of functions like convex and strongly convex functions.

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