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Starlike Functions With Respect to a Boundary Point Connected With Bounded Radius Rotation

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ABSTRACT

In the current article, we introduce a new subclass of univalent function $\mathscr{G}_\mu$ with bounded radius rotation. For this new class, the authors establish many interesting relations between these classes and existing classes. Furthermore, authors try to find coefficient estimations for this new subclass. A few examples of the functions belonging to the classes $\mathscr{G}_\mu$ are also stated.

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1 | Introduction

Indicate $\mathscr{A}$ to be the class of all functions denoted as follows:

$$f(\upsilon) = \upsilon + \sum_{m=2}^{\infty} a_m \upsilon^m, \tag{1}$$

which are holomorphic in the open unit disk $\mathbb{U}_0 := \{\upsilon \in \mathbb{C} : |\upsilon| < 1\}$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all univalent functions. The two subclasses of univalent functions, $\mathscr{S}^*$ and $\mathscr{C}$, which are commonly known as the starlike and convex functions, respectively, are introduced by Robertson [1]. The following are the analytic representations of classes $\mathscr{S}^*$ and $\mathscr{C}$:

$$\mathscr{S}^* := \left\{ f \in \mathscr{A} : \Re\left(\frac{\upsilon f'(\upsilon)}{f(\upsilon)}\right) > 0,\ \upsilon \in \mathbb{U}_0 \right\},$$

and

$$\mathscr{C} := \left\{ f \in \mathscr{A} : \Re\left(1 + \frac{\upsilon f''(\upsilon)}{f'(\upsilon)}\right) > 0, \quad \upsilon \in \mathbb{U}_0 \right\}.$$

Let χ_μ be the class of bounded variation real-valued functions $\chi(\theta)$ on $[0, 2\pi]$ which satisfy the given requirements as follows:

$$\int_0^{2\pi} d\chi(\theta) = 2 \text{ and } \int_0^{2\pi} |d\chi(\theta)| \leq \mu,\ \mu \geq 2.$$

In a given simply connected domain, a function f that is both holomorphic and locally univalent is said to have bounded radius rotation if the total variation of the angle between the radius vector $f(re^{i\theta})$ and the positive real axis is bounded by $\mu\pi$. Let $\mathscr{R}_\mu$ defines the family of functions f represented by Equation (1) satisfy the following:

$$\int_0^{2\pi} \left| \Re\left(\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right) \right| d\theta \leq \mu\pi.$$

Pinchuk [2] gave an integral representation of functions belonging to the class $\mathscr{R}_\mu$ as follows: If a function $f \in \mathscr{R}_\mu$ represented by Equation (1), then f will have an integral representation as follows:

$$\frac{f(\upsilon)}{\upsilon} = \exp\left\{-\int_0^{2\pi} \log|1-\upsilon e^{-i\theta}|d\chi(\theta)\right\}.$$

where $\chi(\theta) \in \chi_\mu$. Moreover, $\mathscr{R}_2$ is the class of normalized starlike functions. The family of functions f defined by Equation (1) that conformally transfer $\mathbb{U}_0$ onto an image domain $f(\mathbb{U}_0)$ of boundary rotation at most $\mu\pi$ is denoted by $\mathscr{V}_\mu$. Since the boundary rotation is the total variation of the argument of the boundary tangent vector, we have the function $f \in \mathscr{V}_\mu$ that satisfies the following integral representation:

$$\int_0^{2\pi} \left|\Re\left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right| d\theta \leq \mu\pi.$$

$\mathscr{V}_2$ is the class of normalized convex functions and for $2 \leq \mu \leq 4$, $\mathscr{V}_\mu$ contains only univalent functions. According to [3], a function $f \in \mathscr{V}_\mu$ if and only if it is as follows:

$$f'(\upsilon) = \exp\left\{-\int_0^{2\pi} \log|1-\upsilon e^{-i\theta}|d\chi(t)\right\},$$

where $\chi(\theta) \in \chi_\mu$. It is clear that for $\mu_1 < \mu_2$, then $\mathscr{V}_{\mu_1} \subset \mathscr{V}_{\mu_2}$. There exists an interesting relation between the classes $\mathscr{R}_\mu$ and $\mathscr{V}_\mu$ as follows:

$$\upsilon f'(\upsilon) \in \mathscr{R}_\mu \text{ if and only if } f(\upsilon) \in \mathscr{V}_\mu. \tag{2}$$

Let $\mathscr{P}_\mu$ defines the family of functions $p(0) = 1$ holomorphic in $\mathbb{U}_0$ and having representation as follows:

$$p(\upsilon) = \frac{1}{2}\int_0^{2\pi} \frac{1+\upsilon e^{-i\theta}}{1-\upsilon e^{-i\theta}} d\chi(\theta),$$

where $\chi(\theta) \in \chi_\mu$ or equivalently, $\mathscr{P}_\mu$ defines the family of functions p holomorphic in $\mathbb{U}_0$, satisfying $p(0) = 1$, and

$$\int_0^{2\pi} |\Re p(re^{i\theta})| d\theta \leq \mu\pi.$$

The class $\mathscr{P}_\mu$ generalizes the class of holomorphic functions with positive real part. $\mathscr{P}_2$ is the class of functions with positive real part. The class $\mathscr{P}_\mu$ is connected with the class $\mathscr{P}$ by the following relation. For $p \in \mathscr{P}_\mu$, we have

$$p(\upsilon) = \left(\frac{\mu}{4} + \frac{1}{2}\right) p_1(\upsilon) - \left(\frac{\mu}{4} - \frac{1}{2}\right) p_2(\upsilon),$$

where $p_1, p_2 \in \mathscr{P}$. A function $u \in \mathscr{P}_\mu(\gamma)$ if and only if there exists a function $q \in \mathscr{P}_\mu$ such that

$$u(\upsilon) = \gamma + (1-\gamma) q(\upsilon). \tag{3}$$

Padmanabhan and Parvatham [4] introduced and investigated the class $\mathscr{P}_\mu(\gamma)$. A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation, which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. For more details about bounded boundary rotation and bounded radius rotation [5–8]. Lehto [9, 10] showed that for a function $f \in \mathscr{V}_\mu$ represented by Equation (1) as follows:

$$|a_2| \leq \frac{\mu}{2} \text{ and } |a_3| \leq \frac{\mu^2 + 2}{6}.$$

In an earlier investigation established by Schiffer and Tammi [11], it is shown that for $f \in \mathscr{V}_\mu$ represented by Equation (1) as follows:

$$|a_4| \leq \frac{\mu^3 + 8\mu}{24}.$$

For E and F, $-1 \leq F < E \leq 1$, a function q analytic in $\mathbb{U}_0$ with $q(0) = 1$ belongs to the class $\mathscr{P}[E, F]$ if

$$q(\upsilon) \prec \frac{1 + E\upsilon}{1 + F\upsilon},$$

where $\prec$ denotes the usual meaning of subordination. The class $\mathscr{P}[E, F]$ introduced by Janowski [12]. An analytic function q with $q(0) = 1$ belongs to the class $\mathscr{P}_\mu[E, F]$ if there exists two functions $q_1(\upsilon)$ and $q_2(\upsilon)$ belongs to the class $\mathscr{P}[E, F]$ such that

$$q(\upsilon) = \frac{\mu + 2}{4} q_1(\upsilon) - \frac{\mu - 2}{4} q_1(\upsilon).$$

Noor [13] studied the classes $\mathscr{V}_\mu[E, F]$ and $\mathscr{R}_\mu[E, F]$. A function f represented by Equation (1) is said to belong to the class $\mathscr{V}_\mu[E, F]$, if

$$1 + \frac{\upsilon f''(\upsilon)}{f'(\upsilon)} \in \mathscr{P}_\mu[E, F].$$

Similarly, a function f represented by Equation (1) is said to belong to the class $\mathscr{R}_\mu[E, F]$, if

$$\frac{\upsilon f'(\upsilon)}{f(\upsilon)} \in \mathscr{P}_\mu[E, F].$$

There exists an interesting relation between the classes $\mathscr{R}_\mu[E, F]$ and $\mathscr{V}_\mu[E, F]$ as follows:

$$\upsilon f'(\upsilon) \in \mathscr{R}_\mu[E, F] \text{ if and only if } f(\upsilon) \in \mathscr{V}_\mu[E, F]. \tag{4}$$

Remark 1.

(i) If we select $E = 1$ and $F = -1$, then the class $\mathscr{P}_\mu[E, F] \equiv \mathscr{P}_\mu[1, -1] \equiv \mathscr{P}_\mu$.

(ii) If we select $E = 1$ and $F = -1$, then the class $\mathscr{V}_\mu[E, F] \equiv \mathscr{V}_\mu[1, -1] \equiv \mathscr{V}_\mu$.

(iii) If we select $E = 1$ and $F = -1$, then the class $\mathscr{R}_\mu[E, F] \equiv \mathscr{R}_\mu[1, -1] \equiv \mathscr{R}_\mu$.

(iv) If we select $E = 1 - 2\gamma$ and $F = -1$, then the class $\mathscr{P}_\mu[E, F] \equiv \mathscr{P}_\mu[1 - 2\gamma, -1] \equiv \mathscr{P}_\mu(\gamma)$.

(v) If we select $E = 1 - 2\gamma$ and $F = -1$, then the class $\mathscr{V}_\mu[E, F] \equiv \mathscr{V}_\mu[1 - 2\gamma, -1] \equiv \mathscr{V}_\mu(\gamma)$.

(vi) If we select $E = 1 - 2\gamma$ and $F = -1$, then the class $\mathscr{R}_\mu[E, F] \equiv \mathscr{R}_\mu[1 - 2\gamma, -1] \equiv \mathscr{R}_\mu(\gamma)$.

Let $\mathscr{G}^*$ be the class of holomorphic functions h in $\mathbb{U}_0$ which satisfy the specified criteria:

(i) h is normalized by the conditions $h(0)=1$ and $h(1)=\lim_{r\to 1}h(r)=0$,

(ii) $h(\mathbb{U}_0)$ lies in a half plane whose boundary passes through the origin, and

(iii) in connection with $h(1), h$ maps $\mathbb{U}_0$ univalently onto a domain that is starlike.

Furthermore, the constant function 1 is also a member of class $\mathscr{G}^*$. Let $\mathscr{G}_*$ be the subset of $\mathscr{G}^*$ for which it is assumed that $h(\upsilon)$ is holomorphic both for $|\upsilon|<1$ and on $|\upsilon|=1$.

Let $\mathscr{G}$ denote the class of functions as follows:

$$h(\upsilon)=1+\sum_{m=0}^{\infty}h_m\upsilon^m=1+h_1\upsilon+h_2\upsilon^2+\cdots \tag{5}$$

holomorphic and nonvanishing in $\mathbb{U}_0$, normalized so that $h(0)=1$ and such that

$$\Re\left(\frac{2\upsilon h'(\upsilon)}{h(\upsilon)}+\frac{1+\upsilon}{1-\upsilon}\right)>0.$$

Robertson [14] showed that $\mathscr{G}_*\subset\mathscr{G}\subset\mathscr{G}^*$ [15].

In the current article, we introduce a new subclass of univalent function $\mathscr{G}_\mu$ with bounded radius rotation. For this new class, the authors drive many interesting relations between $\mathscr{G}_\mu$ and the existing classes that are investigated already. Furthermore, authors find certain coefficient estimation results for the new subclasses. Some examples of the functions belonging to the classes $\mathscr{G}_\mu$ are also stated.

2 | Starlike Function With Respect to Boundary Point With Bounded Radius Rotation

In this section, we define a new subclass of univalent functions, called as starlike function with respect to boundary point with bounded radius rotation.

Definition 1. Let $\mu\geq 2$. Let a function $h(\upsilon)$ represented by Equation (5) is said to be starlike with respect to boundary point with bounded radius rotation if h satisfies the following condition:

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)}-\frac{\upsilon+1}{\upsilon-1}\in\mathscr{P}_\mu,$$

where $\upsilon=\rho e^{i\theta}$. The family of all starlike functions with respect to boundary point with bounded radius rotation is denoted by $\mathscr{G}_\mu$. The class $\mathscr{G}_\mu$ is defined to extend the concept of starlikeness from the interior of the unit disk to functions whose starlike behavior is centered at a boundary point and whose rotation is controlled or bounded. Traditional classes such as starlike or convex functions do not capture the geometric behavior of many analytic mappings that rotate or expand near the boundary. The class $\mathscr{G}_\mu$ allows the function to behave like a starlike mapping while restricting excessive angular distortion. Therefore, $\mathscr{G}_\mu$ provides a more general and flexible framework that includes classical starlike, boundary starlike, and bounded rotation functions as special cases, making it useful for deeper geometric analysis, coefficient estimates, and operator-based studies.

Remark 2. The class of starlike functions with respect to a boundary point and bounded radius rotation represents a balance between freedom and control: the functions allow rotational deformation of the image boundary while ensuring that the mapping retains a global starlike structure and avoids pathological geometric behavior such as sharp spirals, cusps, or folding. This geometric perspective not only highlights the richness of these functions but also explains their relevance in applications involving directional flow, conformal modeling, and boundary-controlled analytic mappings.

Remark 3. If we select $\mu=2$, then the class $\mathscr{G}_\mu\equiv\mathscr{G}_2\equiv\mathscr{G}$, introduce and investigated by Robertson [14].

2.1 | Integral Representation of $\mathscr{G}_\mu$

Theorem 1. *If a function $h\in\mathscr{G}_\mu$, then*

$$h(\upsilon)=(1-\upsilon)\exp\left[-\frac{1}{2}\int_0^{2\pi}\log(1-\upsilon e^{-i\theta})d\chi(\theta)\right], \tag{6}$$

where $\chi(\theta)\in\chi_\mu$.

Proof. Since $h\in\mathscr{G}_\mu$, then $\exists$ an holomorphic function $q(\upsilon)$ belongs to the class $\mathscr{P}_u$ such that

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)}-\frac{\upsilon+1}{\upsilon-1}=q(\upsilon). \tag{7}$$

Equation (7) can be written as follows:

$$\frac{h'(\upsilon)}{h(\upsilon)}-\frac{1}{\upsilon-1}=\frac{1}{2}\left(\frac{q(\upsilon)-1}{\upsilon}\right). \tag{8}$$

Since $q\in\mathscr{P}_u$, according to [3], for a function $q\in\mathscr{P}_u$, there is a function $\chi(t)\in\chi_\mu$ such that

$$q(\upsilon)=\frac{1}{2}\int_0^{2\pi}\frac{1+\upsilon e^{-i\theta}}{1-\upsilon e^{-i\theta}}d\chi(\theta).$$

Therefore,

$$\int_0^{\upsilon}\frac{q(z)-1}{z}dz=-\int_0^{2\pi}\log(1-\upsilon e^{-i\theta})d\chi(\theta). \tag{9}$$

From, Equations (9) and (8) we get Equation (6). This ends the assertion of Theorem 1. ☐

2.2 | Relation Between $\mathscr{G}_\mu$ and $\mathscr{V}_\mu$

Theorem 2. *A function $h\in\mathscr{G}_\mu$ (with $\mu\geq 2$) can be expressed in the form as follows:*

$$h(\upsilon)=(1-\upsilon)\sqrt{g'(\upsilon)}, \tag{10}$$

for some function $g\in\mathscr{V}_\mu$.

Proof. Since $h\in\mathscr{G}_\mu$, then $\exists$ an holomorphic function $q(\upsilon)$ belongs to the class $\mathscr{P}_\mu$ such that

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = q(\upsilon). \tag{11}$$

Since $q \in \mathscr{P}_\mu, \exists\, g \in \mathscr{V}_\mu$ such that

$$q(\upsilon) = 1 + \frac{\upsilon g''(\upsilon)}{g'(\upsilon)}. \tag{12}$$

From, Equations (11) and (12), we get

$$\frac{h'(\upsilon)}{h(\upsilon)} - \frac{1}{\upsilon-1} = \frac{g''(\upsilon)}{2g'(\upsilon)}. \tag{13}$$

Integrating Equation (13) we get Equation (10). This ends the verification of Theorem 2. □

2.3 | Relation Between $\mathscr{G}_\mu$ and $\mathscr{R}_\mu$

Theorem 3. *For $\mu \ge 2$, a function $h \in \mathscr{G}_\mu$ if can be represented in the form as follows:*

$$h(\upsilon) = (1-\upsilon)\sqrt{\frac{g(\upsilon)}{\upsilon}}, \tag{14}$$

where $g \in \mathscr{R}_\mu$.

Proof. From Equation (2) and Theorem 2, we easily get Equation (14). □

2.4 | Relation Between $\mathscr{G}_\mu$ and $\mathscr{K}$

Theorem 4. *Every function $h \in \mathscr{G}_\mu$ may be constructed from two functions $\hbar_1, \hbar_2 \in \mathscr{K}$ through the rule as follows:*

$$h(\upsilon) = (1-\upsilon)\frac{[\hbar_1'(\upsilon)]^{\left(\frac{\mu+2}{8}\right)}}{[\hbar_2'(\upsilon)]^{\left(\frac{\mu-2}{8}\right)}}. \tag{15}$$

Proof. Since $h \in \mathscr{G}_\mu, \exists$ an holomorphic function $q(\upsilon)$ belonging to the class $\mathscr{P}_\mu$ such that

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = q(\upsilon). \tag{16}$$

Since $q \in \mathscr{P}_\mu, \exists\, q_1, q_2 \in \mathscr{P}$ such that

$$q(\upsilon) = \frac{\mu+2}{4} q_1(\upsilon) - \frac{\mu-2}{4} q_2(\upsilon). \tag{17}$$

Since $q_1, q_2 \in \mathscr{P}, \exists\, \hbar_1, \hbar_2 \in \mathscr{K}$ such that

$$q_1(\upsilon) = 1 + \frac{\upsilon \hbar_1''(\upsilon)}{\hbar_1'(\upsilon)} \text{ and } q_2(\upsilon) = 1 + \frac{\upsilon \hbar_2''(\upsilon)}{\hbar_2'(\upsilon)}. \tag{18}$$

Hence, from Equations (16)–(18) we get

$$\frac{h'(\upsilon)}{h(\upsilon)} = \frac{1}{2}\left[\frac{\mu+2}{4}\frac{\hbar_1''(\upsilon)}{\hbar_1'(\upsilon)} - \frac{\mu-2}{4}\frac{\hbar_2''(\upsilon)}{\hbar_2'(\upsilon)}\right]. \tag{19}$$

Integrating Equation (19) we get Equation (15), completing the verification of Theorem 4. □

2.5 | Relation Between $\mathscr{G}_\mu$ and $\mathcal{S}^*$

Theorem 5. *A function $h \in \mathscr{G}_\mu (\mu \ge 2)$ can be represented as follows:*

$$h(\upsilon) = \sqrt{\upsilon}(1-\upsilon)\frac{[\hbar_1(\upsilon)]^{\left(\frac{\mu+2}{8}\right)}}{[\hbar_2(\upsilon)]^{\left(\frac{\mu-2}{8}\right)}}, \tag{20}$$

for certain functions $\hbar_1, \hbar_2 \in \mathcal{S}^*$.

Proof. We know that a function $\varphi \in \mathscr{K}$ if and only if $\upsilon\varphi' \in \mathcal{S}^*$. Similarly if $\varphi \in \mathcal{S}^*$ if and only if

$$g' = \frac{\varphi(\upsilon)}{\upsilon} \in \mathscr{K}.$$

Using Theorem 4 we get Equation (58) finishing the verification of Theorem 5. □

Theorem 6. *For $\mu \ge 2$ and $0 \le \gamma < 1$, a function $h \in \mathscr{G}_\mu$ if*

i. *$\exists\, \hbar_1 \in \mathscr{V}_\mu(\gamma)$ such that*

$$h(\upsilon) = (1-\upsilon)[\hbar_1'(\upsilon)]^{\frac{1}{2(1-\gamma)}}.$$

ii. *$\exists\, \hbar_2 \in \mathscr{R}_\mu(\gamma)$ such that*

$$h(\upsilon) = (1-\upsilon)\left[\frac{\hbar_2'(\upsilon)}{\upsilon}\right]^{\frac{1}{2(1-\gamma)}}.$$

Proof. Using Equations (2) and (3), together with Theorems 2 and 3, the result follows immediately, completing the proof of Theorem 6. □

Theorem 7. *For $\mu \ge 2$, a function $h \in \mathscr{G}_\mu$, then for $|\upsilon| = r < 1$*

$$(1-r)\frac{(1-r)^{\frac{\mu-2}{4}}}{(1+r)^{\frac{\mu+2}{4}}} \le |h(\upsilon)| \le (1-r)\frac{(1+r)^{\frac{\mu-2}{4}}}{(1-r)^{\frac{\mu+2}{4}}}. \tag{21}$$

Proof. By Theorem 1, $h \in \mathscr{G}_\mu$, we have

$$h(\upsilon) = (1-\upsilon)\exp\left[-\frac{1}{2}\int_0^{2\pi} \log(1-\upsilon e^{-i\theta})d\chi(\theta)\right].$$

Let us take $\upsilon = re^{i\theta}$, then

$$|h(re^{it})| \le (1-r)\exp\left[-\frac{1}{2}\int_0^{2\pi} \log|1-\upsilon e^{-i\theta}|d\nu(t)\right].$$

We may write $\chi(\theta) = \chi_1(\theta) - \chi_2(\theta)$ since $\chi(\theta)$ is a nondecreasing function with bounded variation in $[0, 2\pi]$, and both $\chi_1(\theta)$ and $\chi_2(\theta)$ are nondecreasing functions with bounded variation in [0, 2π] and satisfies as follows:

$$\int_0^{2\pi} d\chi_1(\theta) \le \frac{\mu+2}{2} \text{ and } \int_0^{2\pi} d\chi_2(\theta) \le \frac{\mu-2}{2}.$$

Now, we can write it as follows:

$$-\frac{1}{2}\int_0^{2\pi}\log|1-ve^{-i\theta}|d\chi(\theta)=\frac{1}{2}\left[\int_0^{2\pi}\log|1-ve^{-i\theta}|d\chi_2(\theta)-\int_0^{2\pi}\log|1-ve^{-i\theta}|d\chi_1(\theta)\right]$$
$$\leq\frac{\mu-2}{4}\log(1+r)-\frac{\mu+2}{4}\log(1-r)$$
$$=\log\left[\frac{(1+r)^{\frac{\mu-2}{4}}}{(1-r)^{\frac{\mu+2}{4}}}\right].$$

To proved the lower bound it is sufficient to show that as follows:

$$-\int_0^{2\pi}\log|1-ve^{-i\theta}|d\chi(\theta)\geq\log\left[\frac{(1-r)^{\frac{\mu-2}{4}}}{(1+r)^{\frac{\mu+2}{4}}}\right].$$

As $\int_0^{2\pi}|d\chi(\theta)|=\mu$, then $\int_0^{2\pi}|d\chi_1(\theta)|\leq\frac{\mu+2}{2}$ and $\int_0^{2\pi}|d\chi_2(\theta)|\leq\frac{\mu-2}{2}$. Therefore, we have as follows:

$$-\frac{1}{2}\int_0^{2\pi}\log|1-ve^{-i\theta}|d\chi(\theta)\geq\frac{\mu-2}{4}\log(1-r)-\frac{\mu+2}{4}\log(1+r)$$
$$=\log\left[\frac{(1-r)^{\frac{\mu-2}{4}}}{(1+r)^{\frac{\mu+2}{4}}}\right].$$

Hence, we get

$$|h(re^{i\theta})|\geq\log\left[\frac{(1-r)^{\frac{\mu-2}{4}}}{(1+r)^{\frac{\mu+2}{4}}}\right].$$

This ends the verification of Theorem 7. □

Theorem 8. *If a function $h\in\mathscr{G}_\mu$, then the function $H(v)$ defined as follows:*

$$H(v)=(1-v)\frac{h\left(\dfrac{v+a}{1+v\bar{a}}\right)(1-a)}{h'(a)(1+v\bar{a}-v-a)},$$

for some $|a|<1$ belongs to the class $\mathscr{G}_\mu$.

Proof. As $h\in\mathscr{G}_\mu$, from Theorem 2, there exists a function $g\in\mathscr{V}_\mu$ such that

$$h(v)=(1-v)\sqrt{g'(v)}.$$

It is well known [7], that if $g\in\mathscr{V}_\mu$, then the function $G(v)$ defined as follows:

$$G(v)=\frac{g\left(\dfrac{v+\bar{a}}{1+v\bar{a}}\right)-g(a)}{g'(a)(1-|a|^2)},\ |a|<1.$$

also belongs to $\mathscr{V}_\mu$. Hence we get

$$h(v)=(1-v)\frac{\sqrt{g'\left(\dfrac{v+a}{1+v\bar{s}}\right)}}{\sqrt{g'(a)}(1+z\bar{a})}.$$

This ends the verification of Theorem 8. □

Theorem 9. *For $\mu\geq 2$, a function $h\in\mathscr{G}_\mu$, then*

$$|1+h_1|\leq\frac{\mu}{2},\tag{22}$$

$$|1+2h_2-h_1^2|\leq\frac{\mu}{2},\tag{23}$$

and

$$|1+3h_3-3h_1h_2+h_1^3|\leq\frac{\mu}{2}.\tag{24}$$

Proof. Since $h\in\mathscr{G}_\mu$, then $\exists$ an holomorphic function $q(v)$ belonging to the class $\mathscr{P}_\mu$ such that

$$\frac{2vh'(v)}{h(v)}-\frac{v+1}{v-1}=q(v),\tag{25}$$

where

$$q(v)=1+b_1v+b_2v^2+b_3v^3+\cdots.\tag{26}$$

From Equations (25) and (26), we get

$$2+2h_1=b_1,\tag{27}$$

$$2+4h_2-2h_1^2=b_2,\tag{28}$$

and

$$2+6h_3-6h_1h_2+2h_1^3=b_3.\tag{29}$$

Using $|b_m|\leq\mu$ in Equations (27)–(29) we get Equations (22)–(24) respectively that ends the verification of Theorem 9. □

Remark 4. The sharpness of the obtained coefficient bounds remains an open problem.

3 | Illustrations of Functions in the Class $\mathscr{G}_\mu$.

Example 1. The function $h:\mathbb{U}_0\longrightarrow\mathbb{C}$ defined as follows:

$$h(v)=(1-v)\exp\left(-\frac{v}{2}\right).$$

Taking logarithm on both sides and differentiating with respect to v, we get

$$\frac{2vh'(v)}{h(v)}-\frac{v+1}{v-1}=1-v.$$

Since

$$\int_0^{2\pi} \Re(1-\upsilon)dt = \int_0^{2\pi}(1-\cos t)dt = 2\pi \le \mu\pi.$$

Therefore, $1-\upsilon \in \mathscr{P}_\mu$. Thus the function $h \in \mathscr{G}_\mu$.

Example 2. The function $h:\mathbb{U}_0 \longrightarrow \mathbb{C}$ defined as follows:

$$h(\upsilon) = (1-\upsilon)\exp\left(-\frac{\upsilon}{6}\right).$$

Taking logarithm on both sides and differentiating with respect to υ, we get

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = 1 - \frac{\upsilon}{3}.$$

Since

$$\int_0^{2\pi} \Re\left(1-\frac{\upsilon}{3}\right)dt = \int_0^{2\pi}\left(1-\frac{\cos t}{3}\right)dt = 2\pi \le \mu\pi.$$

Therefore, $1-\frac{\upsilon}{3} \in \mathscr{P}_\mu$. Thus the function $h \in \mathscr{G}_\mu$.

Example 3. The function $h:\mathbb{U}_0 \longrightarrow \mathbb{C}$ defined as follows:

$$h(\upsilon) = (1-\upsilon)\exp\left(-\frac{\mu\upsilon}{4}\right).$$

Taking logarithm on both sides and differentiating with respect to υ,

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = 1 - \frac{\mu\upsilon}{2}.$$

Since

$$\int_0^{2\pi} \Re\left(1-\frac{\mu\upsilon}{2}\right)dt = \int_0^{2\pi}\left(1-\frac{\mu\cos t}{2}\right)dt = 2\pi \le \mu\pi.$$

Therefore, $1-\frac{\mu\upsilon}{2} \in \mathscr{P}_\mu$. Thus the function $h \in \mathscr{G}_\mu$.

Example 4. The function $h:\mathbb{U}_0 \longrightarrow \mathbb{C}$ defined as follows:

$$h(\upsilon) = (1-\upsilon)\exp\left(-\frac{\mu\upsilon}{8}\right).$$

Taking logarithm on both sides and differentiating with respect to υ, we get

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = 1 - \frac{\mu\upsilon}{4}.$$

Since

$$\int_0^{2\pi} \Re\left(1-\frac{\mu\upsilon}{4}\right)dt = \int_0^{2\pi}\left(1-\frac{\mu\cos t}{4}\right)dt = 2\pi \le \mu\pi.$$

Therefore, $1-\frac{\mu\upsilon}{4} \in \mathscr{P}_\mu$. Thus the function $h \in \mathscr{G}_\mu$.

4 | Some Classes Are Closely Related to Starlike Function With Respect to Boundary Point With Bounded Radius Rotation

Definition 2. Let $-1 \le F < E \le 1$ and $\mu \ge 2$. A function $f(\upsilon)$ represented by Equation (5) is belongs to the class $\mathscr{G}_\mu[E,F]$ if

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} + \frac{1+E\upsilon}{1+F\upsilon} \in \mathscr{P}_\mu[E,F].$$

Remark 5. If we choose $E=1$ and $F=-1$, then the class $\mathscr{G}_\mu[E,F] \equiv \mathscr{G}_\mu[1,-1] \equiv \mathscr{G}_\mu$, introduced in Definition 1.

Theorem 10. *For $-1 \le F < E \le 1$ and $\mu \ge 2$. If a function $h \in \mathscr{G}_\mu[E,F]$, then*

$$h(\upsilon) = (1+F\upsilon)^{-\frac{E-F}{2F}} \exp\left[\frac{E-F}{4F}\int_0^{2\pi} \log(1+F\upsilon e^{-i\theta})d\chi(\theta)\right], \tag{30}$$

where $\chi(\theta) \in \chi_\mu$.

Proof. Since $h \in \mathscr{G}_\mu[E,F]$, then $\exists$ an holomorphic function $q(\upsilon)$ belongs to the class $\mathscr{P}_\mu[E,F]$ such that

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} + \frac{1+E\upsilon}{1+F\upsilon} = q(\upsilon). \tag{31}$$

Equation (31) can be written as follows:

$$\frac{h'(\upsilon)}{h(\upsilon)} + \frac{E-F}{2}\frac{1}{1+F\upsilon} = \frac{1}{2}\left(\frac{q(\upsilon)-1}{\upsilon}\right). \tag{32}$$

Since $q \in \mathscr{P}_\mu[E,F]$, by representation theorem, for a function $q \in \mathscr{P}_\mu[E,F]$, there is a function $\chi(t) \in \chi_\mu$ such that

$$q(\upsilon) = \frac{1}{2}\int_0^{2\pi} \frac{1+E\upsilon e^{-i\theta}}{1+F\upsilon e^{-i\theta}}d\chi(\theta). \tag{33}$$

Therefore,

$$\int_0^{\upsilon} \frac{q(z)-1}{z}dz = \frac{E-F}{2F}\int_0^{2\pi} \log(1+F\upsilon e^{-i\theta})d\chi(\theta). \tag{34}$$

From, Equations (34) and (32) we get Equation (30). This ends the verification of Theorem 10. □

Theorem 11. *For $-1 \le F < E \le 1$ and $\mu \ge 2$. If a function $h \in \mathscr{G}_\mu[E,F]$ if $\exists$ a function $g_e \in \mathscr{V}_\mu[E,F]$ such that*

$$h(\upsilon) = (1+F\upsilon)^{\frac{F-E}{2F}}\sqrt{g_e'(\upsilon)}. \tag{35}$$

Proof. Since $h \in \mathscr{G}_\mu[E,F]$, then $\exists$ an holomorphic function $q(\upsilon)$ belongs to the class $\mathscr{P}_\mu[E,F]$ such that

$$\frac{2\upsilon h'(\upsilon)}{h(\upsilon)} + \frac{1+E\upsilon}{1+F\upsilon} = q(\upsilon). \tag{36}$$

Since $q \in \mathscr{P}_\mu[E,F]$, $\exists\, g_e \in \mathscr{P}_\mu[E,F]$ such that

$$q(v) = 1 + \frac{v g_e''(v)}{g_e'(v)}. \tag{37}$$

From, Equations (36) and (37), we get

$$\frac{h'(v)}{h(v)} + \frac{E-F}{2F}\frac{1}{1+Fv} = \frac{g''(v)}{2g'(v)}. \tag{38}$$

Integrating Equation (38) we get Equation (36). This ends the verification of Theorem 11. □

Theorem 12. *For $-1 \leq F < E \leq 1$ and $\mu \geq 2$. A function h is said to belong to the class $\mathscr{G}_\mu[E,F]$ provided that there exists a function $g_e \in \mathscr{R}_\mu[E,F]$ such that*

$$h(v) = (1+Fv)^{\frac{F-E}{2F}}\sqrt{\frac{g_e(v)}{v}}. \tag{39}$$

Proof. From Equation (4) and Theorem 11, we easily get Equation (39). □

Let us consider the Pick function as follows:

$$P(v;B) := \frac{4v}{\left(\sqrt{(1-v)^2 + 4v/B} + (1-v)\right)^2}, \quad \sqrt{1} = 1,\ v \in \mathbb{U}_0.$$

By using the Pick function $P(v;B)$ we consider a class which is closely related to the class $\mathscr{G}_\mu$.

Definition 3. Let $B > 1$ and $\mu \geq 2$. A function $h(v)$ represented by Equation (5) is belongs to the class $\mathscr{G}_\mu(B)$ if

$$\frac{2vh'(v)}{h(v)} + \frac{vP'(v;B)}{P(v;B)} \in \mathscr{P}_\mu.$$

Remark 6. If we choose $\mu = 2$, then $\mathscr{G}_\mu(B) \equiv \mathscr{G}_2(B) \equiv \mathscr{G}(B)$ is the class of functions satisfy the condition:

$$\Re\left(\frac{2vh'(v)}{h(v)} + \frac{vP'(v;B)}{P(v;B)}\right) > 0.$$

The class $\mathscr{G}(B)$ introduced by Jakubowsk and Włodarczyk [16].

Theorem 13. *If a function $h \in \mathscr{G}_\mu(B)$, then*

$$h(v) = \sqrt{\frac{v}{P(v;B)}}\exp\left[-\frac{1}{2}\int_0^{2\pi}\log(1 - ve^{-i\theta})d\chi(\theta)\right] \tag{40}$$

where $\chi(\theta) \in \chi_\mu$.

Proof. Since $h \in \mathscr{G}_\mu$, then $\exists$ an holomorphic function $q(v)$ belongs to the class $\mathscr{P}_u$ such that

$$\frac{2vh'(v)}{h(v)} + \frac{vP'(v;B)}{P(v;B)} = q(v). \tag{41}$$

Equation (41) can be written as follows:

$$\frac{h'(v)}{h(v)} + \frac{P'(v;B)}{2P(v;B)} - \frac{1}{2v} = \frac{1}{2}\left(\frac{q(v)-1}{v}\right). \tag{42}$$

From, Equations (9) and (42) we get Equation (40). This ends the verification of Theorem 13. □

Theorem 14. *For $B > 1$ and $\mu \geq 2$, a function $h \in \mathscr{G}_\mu(B)$ if $\exists$ a function $g \in \mathscr{V}_\mu$ such that*

$$h(v) = \sqrt{\frac{vg'(v)}{P(v;B)}}. \tag{43}$$

Proof. Since $h \in \mathscr{G}_\mu(B)$, then $\exists$ an holomorphic function $q(v)$ belongs to the class $\mathscr{P}_\mu$ such that

$$\frac{2vh'(v)}{h(v)} + \frac{vP'(v;B)}{P(v;B)} = q(v). \tag{44}$$

From, Equations (12) and (44), we get

$$\frac{h'(v)}{h(v)} + \frac{P'(v;B)}{2P(v;B)} = \frac{1}{2v} + \frac{g''(\varsigma)}{g'(\varsigma)}. \tag{45}$$

Integrating Equation (45) we get Equation (43). This ends the verification of Theorem 14. □

Theorem 15. *For $B > 1$ and $\mu \geq 2$, a function $h \in \mathscr{G}_\mu(B)$ if $\exists$ a function $g \in \mathscr{R}_\mu$ such that*

$$h(v) = \sqrt{\frac{g(v)}{P(v;B)}}. \tag{46}$$

Proof. From Equation (2) and Theorem 14, we easily get Equation (46). □

5 | Spiral-Like Functions With Respect to a Boundary Point With Bounded Radius Rotation

In this section, we define a new subclass of univalent functions called as spiral-like functions with respect to a boundary point with bounded radius rotation.

Definition 4. Let $\eta \in \mathscr{D} := \{\delta \in \mathbb{C} : |\delta - 1| \leq 1,\ \delta \neq 0\}$ and $\mu \geq 2$. A function $f(v)$ represented by Equation (5) is said to be spiral-like functions with respect to a boundary point with bounded radius rotation if h satisfies the following condition:

$$\frac{2vf'(v)}{\eta f(v)} - \frac{v+1}{v-1} \in \mathscr{P}_\mu.$$

where $v = re^{i\theta}$. The family of all spiral-like functions with respect to boundary point with bounded radius rotation is denoted by $\mathcal{S}_\mu^b(\eta)$.

Remark 7.

i. If we choose $\mu = 2$, then we have $\mathcal{S}^b_\mu(\eta) \equiv \mathcal{S}^b_2(\eta) \equiv \mathcal{S}^b(\eta)$ is the class of functions satisfy the condition:

$$\Re\left(\frac{2\upsilon f'(\upsilon)}{\eta f(\upsilon)} - \frac{\upsilon+1}{\upsilon-1}\right) > 0.$$

The class $\mathcal{S}^b(\eta)$ introduced by Mohd and Darus [17].

ii. If we choose $\eta = 1$, then the class $\mathcal{S}^b(\eta) \equiv \mathcal{S}^b(1) \equiv \mathcal{G}_\mu$, introduced in Definition 1.

5.1 | Integral Representation of $\mathcal{S}^b_\mu(\eta)$

Theorem 16. *If a function $f \in \mathcal{S}^b_\mu(\eta)$, then*

$$[f(\upsilon)]^{\frac{2}{\eta}} = (1-\upsilon)^2 \exp\left\{-\int_0^{2\pi} \log|1-\upsilon e^{-i\theta}| d\chi(\theta)\right\}, \tag{47}$$

where $\chi(\theta) \in \chi_\mu$.

Proof. Since $f \in \mathcal{S}^b_\mu(\eta)$, then $\exists$ an holomorphic function $q(\upsilon)$ belongs to the class $\mathcal{P}_u$ such that

$$\frac{2\upsilon f'(\upsilon)}{\eta f(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = q(\upsilon). \tag{48}$$

We rewrite Equation (48) as follows:

$$\frac{2f'(\upsilon)}{\eta f(\upsilon)} - \frac{2}{\upsilon-1} = \frac{q(\upsilon)-1}{\upsilon}. \tag{49}$$

From, Equations (9) and (49) we get Equation (47). This ends the verification of Theorem 16. □

Theorem 17. *Let a function $f \in \mathcal{S}^b_\mu(\eta)$. Then the function is as follows:*

$$g(\upsilon) = \int_0^\upsilon \frac{[f(\zeta)]^{\frac{2}{\eta}}}{(1-\zeta)^2} d\zeta,$$

belongs to the class $\mathcal{V}_\mu$.

Proof. Since $f \in \mathcal{S}^b_\mu(\eta)$, $\exists$, an holomorphic function $q(\upsilon)$ belonging to the class $\mathcal{P}_u$ such that

$$\frac{2\upsilon f'(\upsilon)}{\eta f(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = q(\upsilon). \tag{50}$$

As $q \in \mathcal{P}_\mu$ then $\exists$ a function $g \in \mathcal{V}_\mu$ such that

$$q(\upsilon) = 1 + \frac{\upsilon g''(\upsilon)}{g'(\upsilon)}. \tag{51}$$

Hence, from Equations (50) and (51), we get

$$\frac{2f'(\upsilon)}{\eta f(\upsilon)} - \frac{2}{\upsilon-1} = \frac{g''(\upsilon)}{g'(\upsilon)}. \tag{52}$$

Equation (52) ends the verification of Theorem 17. □

Theorem 18. *Let a function $f \in \mathcal{S}^b_\mu(\eta)$. Then the function*

$$g(\upsilon) = \frac{\upsilon[f(\upsilon)]^{\frac{2}{\eta}}}{(1-\upsilon)^2},$$

belongs to the class $\mathcal{R}_\mu$.

Proof. From Equation (2) and Theorem 17, we easily prove Theorem 18. □

Theorem 19. *If a function $f \in \mathcal{S}^b_\mu(\eta)$, then $\exists$ two functions g_1, $g_2 \in \mathcal{K}$ such that*

$$[f(\upsilon)]^{\frac{2}{\eta}} = (1-\upsilon)^2 \frac{[g_1'(\upsilon)]^{\left(\frac{\mu+2}{4}\right)}}{[g_2'(\upsilon)]^{\left(\frac{\mu-2}{4}\right)}}. \tag{53}$$

Proof. Since $h \in \mathcal{S}^b_\mu(\eta)$, $\exists$ an holomorphic function $q(\upsilon)$ belonging to the class $\mathcal{P}_\mu$ such that

$$\frac{2\upsilon f'(\upsilon)}{\eta f(\upsilon)} - \frac{\upsilon+1}{\upsilon-1} = q(\upsilon). \tag{54}$$

Since $q \in \mathcal{P}_\mu$, $\exists q_1, q_2 \in \mathcal{P}$ such that

$$q(\upsilon) = \frac{\mu+2}{4} q_1(\upsilon) - \frac{\mu-2}{4} q_2(\upsilon). \tag{55}$$

Since $q_1, q_2 \in \mathcal{P}$, $\exists g_1, g_2 \in \mathcal{K}$ such that

$$q_1(\upsilon) = 1 + \frac{\upsilon g_1''(\upsilon)}{g_1'(\upsilon)} \text{ and } q_2(\upsilon) = 1 + \frac{\upsilon g_2''(\upsilon)}{g_2'(\upsilon)}. \tag{56}$$

Hence, from Equations (54) and (55) and Equation (56) we get

$$\frac{h'(\upsilon)}{h(\upsilon)} = \frac{1}{2}\left[\frac{\mu+2}{4}\frac{g_1''(\upsilon)}{g_1'(\upsilon)} - \frac{\mu-2}{4}\frac{g_2''(\upsilon)}{g_2'(\upsilon)}\right]. \tag{57}$$

Integrating Equation (57) we get Equation (53), completing the verification of Theorem 19. □

Theorem 20. *Let $\mu \geq 2$. A function $f \in \mathcal{S}^b_\mu(\eta)$ if $\exists \hbar_1, \hbar_2 \in \mathcal{S}^*$ such that*

$$[f(\upsilon)]^{\frac{2}{\eta}} = \upsilon(1-\upsilon)^2 \frac{[\hbar_1(\upsilon)]^{\left(\frac{\mu+2}{8}\right)}}{[\hbar_2(\upsilon)]^{\left(\frac{\mu-2}{8}\right)}}. \tag{58}$$

Proof. From Equation (2) and Theorem 19, we easily get Equation (58). □

Theorem 21. *For $\mu \in [2, \infty)$, $\gamma \in [0, 1)$, a function $f \in \mathcal{S}^b_\mu(\eta)$ if*

i. *$\exists \hbar_1 \in \mathcal{V}_\mu(\gamma)$ such that*

$$h(\upsilon) = (1-\upsilon)^{\frac{1}{\eta}} [\hbar_1'(\upsilon)]^{\frac{\eta}{2(1-\gamma)}}.$$

ii. *$\exists \hbar_2 \in \mathcal{R}_\mu(\gamma)$ such that*

$$h(\upsilon) = (1-\upsilon)^{\frac{1}{\eta}} \left[\frac{\hbar_2'(\upsilon)}{\upsilon}\right]^{\frac{\eta}{2(1-\gamma)}}.$$

Proof. Using Equations (2) and (3), together with Theorems 17 and 18, the result follows immediately, completing the proof of Theorem 21. □

6 | Conclusion

In the investigation carried out, the authors proposed a subclass $\mathscr{G}_{\mu}$, consisting of univalent functions with bounded radius rotation. A motivating connection between the new subclass with the other existing classes also derived. The coefficient bounds for this new subclass are also derived. Some classes which are strongly related to the class of starlike functions with respect to boundary points with bounded radius rotation are also derived. Many appealing examples are also formed for the class $\mathscr{G}_{\mu}$.

The results obtained in this study open various possible directions for further investigation. One natural extension is to explore sharp coefficient problems and extremal functions associated with the class $\mathscr{G}_{\mu}$. Another promising direction is to examine convolution (Hadamard product) properties, differential subordination relations, and inclusion criteria with other well-established subclasses of univalent and multivalent functions.

In addition, it would be of interest to study growth, distortion, and covering theorems, as well as radius problems such as radius of starlikeness, convexity, and close-to-convexity for the class $\mathscr{G}_{\mu}$. The behavior of integral and differential operators acting on $\mathscr{G}_{\mu}$ also remains an open area for exploration.

Finally, potential applications of the class $\mathscr{G}_{\mu}$ in geometric function theory, fractional calculus, and $q-$ analysis may lead to further significant developments. These directions will be addressed in future work.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

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