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On stationary solutions and inviscid limits for generalized Constantin–Lax–Majda equation with $O(1)$ forcing

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Abstract

The generalized Constantin–Lax–Majda (gCLM) equation was introduced to model the competing effects of advection and vortex stretching in hydrodynamics. Recent investigations revealed possible connections with the two-dimensional turbulence. With this connection in mind, we consider the steady problem for the viscous gCLM equations on $\mathbb{T}$: $av\omega_x - v_x\omega = \nu\Delta\omega + f$, $v = (-\Delta)^{-\frac{1}{2}}\omega$, where $a \in \mathbb{R}$ is the parameter measuring the relative strength between advection and stretching, $\nu > 0$ is the viscosity constant, and f is a given $O(1)$ -forcing independent of ν . For some range of parameters, we establish existence and uniqueness of stationary solutions. We then numerically investigate the behaviour of solutions in the vanishing viscosity limit, where bifurcations appear, and new solutions emerge. When the parameter a is away from $[-1/2, 1]$, we verify that there is convergence towards smooth stationary solutions for the corresponding inviscid equation. Moreover, we analyse the inviscid limit in the fractionally dissipative case, as well as the behaviour of singular limiting solutions.

Keywords: fluid model equations, vanishing viscosity limit, steady-state solutions, existence, bifurcation, asymptotics,

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(Some figures may appear in colour only in the online journal)

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1. Introduction

In this study, we consider an inviscid limit problem for the generalized Constantin–Lax–Majda (CLM) equation with $O(1)$ forcing. Specifically, we are interested in the limit of stable stationary solutions as the viscosity constant tends to zero. To be concrete, let us begin with recalling the generalized CLM equation:

$$\partial_t \omega + a v \omega_x - v_x \omega = -\nu (-\Delta)^\beta \omega + f, \quad v = (-\Delta)^{-\frac{1}{2}} \omega. \quad (1.1)$$

We fix the spatial domain to be $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. In the aforementioned equation, $a \in \mathbb{R}$, $\beta > 0$, $\nu > 0$, and $f : \mathbb{T} \rightarrow \mathbb{R}$ are parameters defining the equation; a determines the relative strength between advection and stretching terms, $\nu > 0$ represents the viscosity constant, $\beta > 0$ determines the order of dissipation, and f denotes a time-independent external force. Without loss of generality, we shall assume that ω and f are mean zero. For a mean-zero function g , the fractional Laplacian of order β is defined by

$$g(x) = \sum_{k \in \mathbb{Z}} g_k e^{ikx}, \quad ((-\Delta)^\beta g)(x) = \sum_{k \in \mathbb{Z}} |k|^{2\beta} g_k e^{ikx}.$$

We are specifically interested in the inviscid limit problem for stationary solutions: given a , β , and f , we consider the set of time-independent solutions $\{\bar{\omega}^{(\nu)}\}$ for each $\nu > 0$, and investigate the limit as $\nu \rightarrow 0$. We would like to answer the following question: under which circumstances is there a branch of stationary solutions $\bar{\omega}^{(\nu)}$ that converges to a stationary solution $\bar{\omega}$ to the inviscid ($\nu = 0$) equation?

1.1. Motivation

Let us present some motivations for considering the family of generalized CLM equation in the first place, which help us in putting our work into context. Then, let us explain the motivation for studying the specific problem of inviscid limit of stationary solution sequences.

History and motivation for gCLM. The Constantin–Lax–Majda equation,

$$\partial_t \omega = \omega H[\omega], \quad \omega : \mathbb{R} \rightarrow \mathbb{R},$$

was introduced in [10] as a formal 1D toy model to study the effect of the vortex stretching term in the 3D incompressible Euler equations. The viscous counterpart was investigated by Schochet [51]. De Gregorio [18, 19] then suggested to include a transport term to transform the equation to a more realistic model:

$$\partial_t \omega + v \partial_x \omega = \partial_x v \omega, \quad \partial_x v = H[\omega]. \quad (1.2)$$

The model incorporating parameter a was introduced by Okamoto *et al* [47]; this parameter represents the relative strength between advection and stretching terms. When $a \leq 0$, the transport term is either absent or work together with the stretching towards growth of $\|\omega\|_{L^\infty}$. On the other hand, for $a > 0$, the transport term competes with the stretching, the extreme case being the ‘ $a = \infty$ ’ case considered in [49]:

$$\partial_t \omega + v \partial_x \omega = 0, \quad \partial_x v = H[\omega]. \quad (1.3)$$

Different values of a appear from various systems arising in fluid dynamics. In the case $a = -1$, one can define the anti-derivative of ω to be Θ and obtain

$$\partial_t \Theta + v \partial_x \Theta = 0,$$

which has been suggested by Cordoba *et al* in [17] as a toy model for the surface quasi-geostrophic (SQG) equation. To explain further, let us recall the SQG equation (defined either on $\mathbb{T}^2$ or $\mathbb{R}^2$):

$$\partial_t \Theta + u \cdot \nabla \Theta = 0, \quad u = \nabla^\perp (-\Delta)^{-1} \Theta. \quad (1.4)$$

In [4], the authors realized that by inserting the so-called *stagnation point similitude* ansatz

$$\Theta(t, x_1, x_2) = x_2 \omega(t, x_1),$$

the SQG equation reduces to the De Gregorio equation (1.2). Similarly, in [23], it was found that by taking *radial homogeneous* ansatz (in polar coordinates)

$$\Theta(t, r, \theta) = r \omega(t, \theta), \quad \theta \in [-\pi, \pi),$$

(1.4) reduces to (1.3) with parameter $a = 2$, up to a lower-order term. Applying the aforementioned ansatz to the so-called α -SQG equation, one can obtain (1.3) with a lying on some interval in $\mathbb{R}$ (see [4, 23] for details). From this point of view, the generalized CLM equation is comparable with the so-called generalized Proudman–Johnson (gPJ) equation, which is obtained via the plugging in of special ansatz to high-dimensional axisymmetric Navier–Stokes equations. Finally, Elgindi [21] noted that after several reductions, the axisymmetric 3D Euler equations resembles the gCLM equation with $a = 3$.

Now, the viscosity term in (1.1) could be added to model viscous fluid systems. We remark that, especially at the level of formal modelling, there is no obvious reason to insist on the usual dissipation term $\beta = 1$ over other values of $\beta > 0$. For example, for the SQG equation, the physically relevant dissipation term is given by the half-Laplacian ($\beta = \frac{1}{2}$); see [12, 13].

Whereas these types of equations were mainly considered simply as formal toy models for fluid systems, as it turns out, there are more merits to study them. First, the solutions of these 1D equations give rise to solutions to actual fluid systems; to explain further,

Recently, Matsumoto and Sakajo ([44, 45]) also numerically investigated the generalized CLM equation involving a viscosity term for $a \leq -1$ (with special emphasis on the $a = -2$ case) and observed phenomena analogous to 2D turbulence. The case $a \leq -1$ is distinguished from $a > -1$ because, in the former, we have the conservation (in the inviscid case) of the following quantity in time:

$$\int_{\mathbb{T}} |\omega|^{-a}(t, x) dx,$$

which the authors considered as an analogue of enstrophy, which is conserved in 2D inviscid flow. We note that in [44], it was found that with deterministic forcing, solutions to (1.1) converge to a stationary one. This conclusion provides the basis of our current study.

Inviscid limit problem. Our main motivation comes from understanding the long-time dynamics of incompressible fluid models, most notably the 2D Navier–Stokes equations (defined on $\mathbb{T}^2 = [-\pi, \pi)^2$):

$$\partial_t u^{(\nu)} + u^{(\nu)} \cdot \nabla u^{(\nu)} + \nabla p = \nu \Delta u^{(\nu)} + f, \quad \nabla \cdot u^{(\nu)} = 0. \quad (1.5)$$

For any $\nu > 0$, the system (1.5) is globally well-posed for smooth initial data, and it is natural to study the set of possible inviscid limits for long-time averages:

$$\lim_{\nu \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u^{(\nu)}(t) dt.$$

We are particularly interested in whether the aforementioned is connected with the corresponding limit for the *inviscid* problem:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(t) dt,$$

where $u(t)$ is a solution of (1.5) with $\nu = 0$, namely the 2D incompressible Euler equations. One can consider the same question for other fluid systems that are globally well-posed; another notable example is the so-called critically dissipative SQG equation on $\mathbb{T}^2$:

$$\partial_t \Theta^{(\nu)} + u^{(\nu)} \cdot \nabla \Theta^{(\nu)} = -\nu(-\Delta)^{\frac{1}{2}} \Theta^{(\nu)} + f, \quad u^{(\nu)} = \nabla^\perp (-\Delta)^{-\frac{1}{2}} \Theta^{(\nu)};$$

see [3, 5, 14, 15, 38]. We mention that the authors in [14] considered the forced, critically dissipative SQG equation and proved the existence of a finite-dimensional global attractor.

We refer to review papers [11, 24] for further discussions and references regarding the problem of long-time inviscid limit. In the absence of physical boundaries, the inviscid limit is relatively well-understood for finite time horizons ([6, 40, 42, 43, 52]), but the long-time problem remains largely open, even from a computational perspective. However, we mention an important paper of Gallet–Young [26] where the authors consider this exact problem on $[0, \pi]^2$ with time-independent forcing $\nabla \times f = \sin(nx_1) \sin(nx_2)$ with some even $n \geq 2$. The authors show, via asymptotic analysis, emergence of a steady vortex condensate concentrated on the lowest mode $\sin(x_1) \sin(x_2)$ whose amplitude is independent of ν as $\nu \rightarrow 0$. The authors emphasize that while the inviscid problem (forced 2D Euler) has a continuous family of vortex condensate states, only a single one is selected in the inviscid limit. In this sense, the dissipative mechanism is remembered in the inviscid equation. They even proceed to develop the theory of selection for general body forces; a system of 3 coupled ODEs mimicking fluid nonlinearity in Fourier space is extracted, which clearly exhibits transference of stability from an $O(\nu^{-1})$ steady state to the others of size $O(1)$. Regarding the vortex condensate, we refer the interested readers to earlier works ([9, 53]).

These phenomena appearing in the inviscid limit are exactly the problems we would like to understand, while employing 1D toy models (specifically, the generalized CLM equation). However, as we shall demonstrate in this paper, even in this 1D setting, the answer to this question is far from being trivial. We emphasize that the long-time average is taken before the inviscid limit (which is physically relevant, as argued in [11]), and the forcing is taken to be *independent* of the viscosity. In principle, if the forcing is $O(1)$ of the viscosity, the solutions $u^{(\nu)}(t)$ for some fixed $t > 0$ may blow up with $\nu \rightarrow 0$. This has been shown to occur when f is ‘mostly’ supported on the lowest Fourier mode ([41]), given that there is a unique stationary solution for each $\nu > 0$ of size $O(\nu^{-1})$ that is globally attracting. However, for general forcing, it is possible that stationary solutions of size $O(\nu^{-1})$ lose stability as ν approaches zero, and that stability is transferred to a new branch of steady states, which is uniformly bounded as $\nu \rightarrow 0$ (see [31, 54]). (Even in situations with unique stable steady state for each $\nu > 0$, it is possible that the limit $\nu \rightarrow 0$ is well-defined in weak topologies and corresponds to a weak steady state of the inviscid problem. This phenomenon was rigorously proved for the 1D Burgers equation in [28]. The case of general Hamilton–Jacobi equations was covered by [20]).

To arrange for such a phenomenon, one may start with a pair $(\bar{u}, f)$ solving

$$\bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} = f, \quad \nabla \cdot \bar{u} = 0,$$

and ask whether there is a sequence of steady states $\bar{u}^{(\nu)}$ (with the same f) converging to $\bar{u}$. We emphasize again that the existence of such a sequence implies that a bifurcation must occur at

some $\nu^* > 0$, because at least when $\nu \gg 1$, there is only one steady state, and that state has size $O(\nu^{-1})$. (These facts are consequences of the Leray–Schauder theorem.) In addition, if for all sufficiently small $\nu > 0$, $\bar{u}^{(\nu)}$ is attracting, then it indicates that at least for some class of initial data, the infinite-time inviscid limit holds true for the forced problem.

Let us mention that another motivation for studying the problem of long-time inviscid limit comes from a study on Hamilton–Jacobi equations, specifically from the issue of selecting natural weak solutions after shock formation. We refer interested readers to the introduction of [29].

1.2. Previous works

The family of equations (1.1) and their variants have been studied intensively in the past decade, from various points of view. We shall restrict ourselves to works directly relevant to ours.

Generalized Proudman–Johnson equation. Recall that the gPJ on $\mathbb{T}$ has the form

$$\partial_t \psi_{xx} + \psi \psi_{xxx} - a \psi_x \psi_{xx} = \nu \psi_{xxxx} + f, \quad a \in \mathbb{R}, \nu > 0. \quad (1.6)$$

For various values of a , (1.6) includes the Burgers, 2D and 3D Navier–Stokes (under certain ansatz), and Hunter–Saxton equations as some special cases. In our previous paper [29], we considered the same problem for the family of gPJ equations; specifically, we considered the sequence of stationary solutions satisfying

$$\psi \psi_{xxx} - a \psi_x \psi_{xx} = \nu \psi_{xxxx} + \frac{a-1}{2} \sin 2x. \quad (1.7)$$

The specific form of forcing in (1.7) is chosen such that in the inviscid case $\nu = 0$, we have $\bar{\psi} = \sin x$ as a solution to

$$\bar{\psi} \bar{\psi}_{xxx} - a \bar{\psi}_x \bar{\psi}_{xx} = \frac{a-1}{2} \sin 2x,$$

whereas for $\nu \gg 1$, the unique steady state for (1.7) is globally attracting and has the form

$$\psi \approx \frac{1}{\nu} \frac{a-1}{32} \sin 2x.$$

It is possible that as $\nu \rightarrow 0$, there will be a bifurcation generating new stable steady states whose inviscid limit is given by $\bar{\psi}$. We have indeed verified numerically in [29] that the bifurcated solutions for some range of a converge to $\sin x$ in the inviscid limit. We also mention works by Kim and Okamoto [35, 36], where the authors studied stationary solutions for the gPJ equation with forcing proportional to the viscosity constant.

Global well-posedness. From an analytical point of view, a recent work by Bae *et al* [2] is the most relevant to ours. Whereas they also considered other types of 1D model equations, restricting their results to ones concerning (1.1), they proved global well-posedness for smooth data when $a \in [-\infty, -2) \cup \{-\frac{1}{2}\}$, and $\beta > \frac{1}{2}$. The case $a = -\frac{1}{2}$ is special, because there is an *a priori* control of the L^2 -norm of v (see the proof of theorem 2, given in a later section). One can refer to the references section in [2] for an extensive list of previous works on global regularity and finite-time singularity formation of various 1D models.

We mention that, very recently, Chen [7] proved global existence in $\mathbb{R}$ for $a < -1$ with critical and supercritical dissipation. The proof can be adapted to the case of $\mathbb{T}$ without much difficulty; mainly, one just needs to extend the key lemma 2.2 to the periodic case, which

we explain in the remark following the proof of theorem 2. We state the main global well-posedness result of [2, 7], as follows:

Theorem 1. Consider (1.1) where the pair (β, a) satisfies one of the following:

- (a) $\beta \geq |a|^{-1}$ and $a \leq -1$,
- (b) $a < -1$ and $|a|^{-1} > \beta > \beta_0$ for some $\beta_0 = \beta_0(a, \|\omega_0\|_{H^1}) < |a|^{-1}$,
- (c) $\beta > \frac{1}{2}$ and $a = -\frac{1}{2}$.

Thus, for any $f \in H^1(\mathbb{T})$ and $\omega_0 \in H^1(\mathbb{T})$, there exists a unique global solution of (1.1) that belongs to $C_{\text{loc}}([0, \infty); H^1(\mathbb{T}))$.

The question of global regularity in the case $a > 0$ remains largely open, because of the absence of coercive conservation laws. A remarkable progress towards this was made in [30] (see also [39]), which established global nonlinear stability of the ‘ground state’ $\bar{\omega} = \sin x$ in the $a = 1, \nu = 0, f = 0$ case. It is possible that the result holds true also for $\sin 2x$, which will be more relevant to our problem. For some recent finite time singularity formation results, see [7, 8, 22] and the references therein. We just remark that for $a = 0$, there is a singularity formation [50] with any order of the viscosity term, and thus it is expected that for small $|a|$, a singularity formation persists. This suggests that the case of small $|a|$ is not an appropriate set-up for the study of inviscid limit; see our numerical results in later sections, which support this idea. In this case our numerical computation blows up for rather moderate R and there is no detection of Hopf bifurcation for time-periodic solution on the continuation branch. In [54], the generation of time-periodic solution due to the special form of 2D Navier–Stokes is confirmed, but we will briefly show in remark 2.7 that this is not the case for the gCLM equation when $a = 0$.

Steady states and travelling wave solutions. Regarding the unforced and inviscid ($f = 0$ and $\nu = 0$) case of (1.1), Okamoto *et al* [48] provided numerical evidences for the existence of non-trivial steady and travelling-wave solutions for any $a > 0$. It is expected that they also persist in the forced case. This is interesting because whereas the problem of global regularity versus finite-time blow up is open for $a > 0$ in general, it shows that there exist at least special solutions that are globally regular. Thus, for the same initial data, the inviscid limit could hold true. We also mention a very recent work by Kim *et al* [33].

1.3. Formulation of problem and summary of results

Let us state precisely the problem that we address in this paper. Given parameters $a \in \mathbb{R}, \beta > 0$, we consider solutions of the following equation:

$$av\omega_x - v_x\omega = -\nu(-\Delta)^\beta\omega + \frac{1-a}{2}\sin 2x. \quad (1.8)$$

Note that we have fixed the forcing to be exactly $\frac{1-a}{2}\sin 2x$. The Fourier mode $\sin 2x$ could be considered as the simplest non-trivial forcing, because in [44, 45] it was demonstrated that with $\sin x$ forcing (and $a < 0$), there is *uniqueness* of steady states, without any bifurcations. Moreover, the coefficient $\frac{1-a}{2}$ was inserted simply to make sure that in the inviscid case, $\bar{\omega} = \sin x$ provides a solution to (1.8). Regarding the parameter β , we shall be concerned mainly with the range $\frac{1}{2} \leq \beta \leq 1$, with particular emphasis on endpoints $\beta = \{\frac{1}{2}, 1\}$, which seem physically relevant. Moreover, one can see that $\beta = \frac{1}{2}$ is critical in the sense that loss of one derivative in the term $v\omega_x$ is exactly balanced by dissipation.

In the subsequent sections, we will investigate both the qualitative and quantitative aspects of the aforementioned problem. More precisely, we would like to establish rigorous analytic

results, such as the existence (or nonexistence), uniqueness, and multiplicity of solutions. Meanwhile, as we face the difficulty of answering the questions for all ranges of a and β , we would like to try some alternative approaches. This is how we adopt some accurate numerical computations and proper asymptotic arguments in later sections. Using these alternative approaches, we found several interesting results on some quantitative features of the solutions, such as bifurcations of solutions and asymptotics in the singular limit.

2. Existence, nonexistence, and uniqueness of solution

2.1. Existence and uniqueness of stationary solutions

In this section, we consider the stationary equation

$$av\omega_x - v_x\omega = -\nu(-\Delta)^\beta\omega + f, \quad v = (-\Delta)^{-\frac{1}{2}}\omega, \quad (2.1)$$

under the simplifying assumption that ω (and also f) is represented by a sine series (i.e., odd function at 0):

$$\omega(x) = \sum_{n=1}^{\infty} b(n) \sin nx.$$

Thus, we have

$$v(x) = -\sum_{n=1}^{\infty} \frac{b(n)}{n} \sin nx.$$

Given $a \in \mathbb{R}$, $\beta > \frac{1}{2}$, and an odd function f , we rewrite (2.1) in the form

$$\omega = -\frac{1}{\nu}(-\Delta)^{-\beta}(av\omega_x - v_x\omega - f). \quad (2.2)$$

We define the right-hand side of (2.2) as an operator $\omega \mapsto T_{\beta,a,f}[\omega]$. When ω is sufficiently smooth (e.g., belongs to $H^s(\mathbb{T})$ with large s), it is not difficult to show that T defines a compact operator from H^s onto itself. In the following lemma, we establish that, indeed, T defines a compact operator in $L^p(\mathbb{T})$ with $p > 1$. (In this section, we shall abuse notation and set L^p to contain only odd functions.)

Lemma 2.1. *Assume that $f \in L^2(\mathbb{T})$ is odd, and assume in addition either one of the following:*

- (a) $\beta > \frac{1}{2}$ and $p \geq \frac{4}{3}$.
- (b) $\beta > \frac{1}{2}$, $\frac{4}{3} > p > 1$, and $a = -1$.
- (c) $\beta > \frac{3}{4}$ and $\frac{4}{3} > p > 1$.

Thus, T continuously extends to a compact operator in $L^p(\mathbb{T})$.

Proof. We rewrite

$$av\omega_x - v_x\omega = a(v\omega)_x - (1+a)v_x\omega.$$

It suffices to show, assuming either one of the conditions in the lemma, that

$$\omega \mapsto a(-\Delta)^{-\beta}\partial_x(v\omega), \quad \omega \mapsto (1+a)(-\Delta)^{-\beta}(v_x\omega)$$

are compact operators. To begin with, recall the classical L^p -estimate

$$\|v_x\|_{L^p(\mathbb{T})} \leq C_p \|\omega\|_{L^p(\mathbb{T})}, \quad 1 < p < \infty.$$

Note that for $\omega \in L^p$, we have $v \in W^{1,p} \subset L^\infty$, and hence $v\omega \in L^p$. (In the endpoint case $p = +\infty$, we instead use the fact that $v \in W^{1,q}$ for any $q < +\infty$.) Given that for $\beta > \frac{1}{2}$, $\omega \mapsto (-\Delta)^{-\beta} \partial_x(v\omega)$ defines a compact operator from L^p to itself (see [46, section 7], for instance), we deduce that $\omega \mapsto (-\Delta)^{-\beta} \partial_x(v\omega)$ is also a compact operator. In the special case $a = -1$, this is the only thing we need to show, which concludes the proof in this case.

We proceed assuming $a \neq -1$. We have $v_x \in L^p$, and then $v_x\omega \in L^{p/2}$. If $p \geq 2$, then $(-\Delta)^{-\beta}(v_x\omega)$ belongs to C^α with $\alpha = \frac{\beta-1}{2}$, which is compactly embedded in L^p for any $p \leq +\infty$. In the case $1 < p < 2$, we instead observe directly that (for $\omega \in H^1$, say) the product $v_x\omega$ has the Fourier series

$$(v_x\omega)(x) = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} b(k)b(n-k) \right) \sin nx,$$

where $b(n)$ is the n th coefficient of the sine series for ω . Clearly,

$$|b(n)| \leq C \|\omega\|_{L^p}, \quad \forall n \geq 1,$$

and from the Hausdorff–Young inequality, we obtain

$$\|\{b\}\|_{\ell^q} \leq C_p \|\omega\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Hence for $p \geq \frac{4}{3}$, we have $\{b\}_{n=1}^\infty \in \ell^4$. From the Plancherel theorem, $(-\Delta)^{-\beta}(v_x\omega)$ belongs to L^2 if and only if

$$\sum_n \frac{1}{n^{4\beta}} \left(\sum_{k=1}^{n-1} b(k)b(n-k) \right)^2 < +\infty,$$

and we may estimate the aforementioned as

$$\begin{aligned} \sum_n \frac{1}{n^{4\beta}} \left(\sum_{k=1}^{n-1} b(k)b(n-k) \right)^2 &\leq \sum_n \frac{1}{n^{4\beta}} \sum_{m=1}^{n-1} \sum_{k=1}^{n-1} b(k)^2 b(m)^2 \leq \sum_n \frac{2n}{n^{4\beta}} \sum_{k=1}^{n-1} b(k)^4 \\ &\leq C \|b\|_{\ell^4}^4 \end{aligned}$$

for $\beta > \frac{1}{2}$. The same argument shows that $(-\Delta)^{-\beta}(v_x\omega)$ belongs to H^s , with $s = \frac{1}{2}(\beta - \frac{1}{2})$. This concludes the lemma under the first assumption.

To finish the proof, we consider the third assumption: with $\beta > \frac{3}{4}$, we see that the n th Fourier coefficient of $(-\Delta)^{-\beta}(v_x\omega)$ satisfies

$$\frac{1}{n^{2\beta}} \left| \sum_{k=1}^{n-1} b(k)b(n-k) \right| \leq \frac{n}{n^{2\beta}} \|\{b\}\|_{\ell^\infty}^2 \leq C \|\omega\|_{L^p}^2 n^{1-2\beta}.$$

We conclude that for $\beta > \frac{3}{4}$, the coefficients are square-summable, and hence $(-\Delta)^{-\beta}(v_x\omega)$ belongs to $H^s(\mathbb{T})$, with $s = \frac{1}{2}(\beta - \frac{3}{4})$. This allows one to define the operator $\omega \mapsto (-\Delta)^{-\beta}(v_x\omega)$ from L^p to $H^s \subset L^p$ even in the range $1 < p < \frac{4}{3}$. $\square$

Theorem 2. Consider (2.1) where the pair (β, a) satisfies one of the following:

- (a) $\beta > \frac{1}{2}$ and $a < -\frac{4}{3}$ or $\beta > \frac{3}{4}$ and $a < -1$,
 (b) $\beta > \frac{1}{2}$ and $a = -\frac{1}{2}$,
 (c) $\beta \geq \frac{1}{2}$, $a \in \mathbb{R}$, and $\nu = \nu(a, \beta, f) > 0$ large.

Thus, for any $f \in H^m(\mathbb{T})$ with $m \geq 1$, there exists a stationary solution $\omega \in H^m$ to (2.1) with any $\nu > 0$.

In the cases (a) and (b), the solution is unique if $\nu > 0$ is sufficiently large, depending on a, β , and f . In the case (c), there exists a constant C^* depending only on the value of a , such that the solution is unique upon the assumption that $\|\omega\|_{H^1} \leq C^*\nu$.

Proof. We shall recall the Leray–Schauder theorem: let X be a Banach space and $T : X \rightarrow X$ be a bounded and compact operator. If there exists $M > 0$ such that any solution to

$$g = \lambda T[g], \quad 0 \leq \lambda \leq 1$$

satisfies $\|g\|_X \leq M$, then there exists a solution to the equation $g = T[g]$.

Case (i): $a \in [-\infty, -1)$. We shall apply the Leray–Schauder theorem with $X = L^p(\mathbb{T})$ where $p = -a$. To do this, we need an *a priori* estimate for solutions to

$$av\omega_x - v_x\omega = -\frac{\nu}{\lambda}(-\Delta)^\beta\omega + f$$

that is uniform for $0 < \lambda \leq 1$. By multiplying both sides of the equation by $\text{sgn}(\omega)|\omega|^{p-1}$ and integrating over $\mathbb{T}$, we have the two terms on the left-hand side cancel each other. To obtain a lower bound on the nonlinear term on the right-hand side, we need the following inequality from Chen:

Lemma 2.2 ([7, proposition 4.1]). For $0 \leq \beta \leq 1$ and $p \geq 1$,

$$\int_{\mathbb{T}} \text{sgn}(g)|g|^{p-1}(-\Delta)^\beta g \, dx \geq C \min \left\{ p-1, \frac{1}{p} \right\} \int_{\mathbb{T}} ((-\Delta)^{\frac{\beta}{2}} |g|^{\frac{p}{2}})^2 \, dx.$$

With this lemma, we obtain

$$\frac{C_p \nu}{\lambda} \int_{\mathbb{T}} \left((-\Delta)^{\frac{\beta}{2}} |\omega|^{\frac{p}{2}} \right)^2 \, dx \leq \|\omega\|_{L^p}^{p-1} \|f\|_{L^p}.$$

However, from Poincaré inequality, we have

$$\|\omega\|_{L^p}^p \leq \frac{\lambda}{C_p \nu} \|\omega\|_{L^p}^{p-1} \|f\|_{L^p},$$

which is the desired *a priori* L^p bound. From lemma 2.1, we determine that the operator

$$\omega \mapsto -\frac{1}{\nu}(-\Delta)^\beta(av\omega_x - v_x\omega - f)$$

is compact in L^p with $p = -a$, upon assuming either $\beta > \frac{1}{2}$ and $a < -\frac{4}{3}$, or $\beta > \frac{3}{4}$ and $a < -1$. Leray–Schauder theorem then produces a stationary solution $\omega \in L^p$. A simple bootstrapping argument shows that this ω actually belongs to H^m , as long as $f \in H^m$ for $m \geq 1$.

Case (ii): $a = -\frac{1}{2}$. In this case, we can show an *a priori* bound for $\|\omega\|_{L^2}$. To see this, multiply both sides of

$$-\frac{1}{2}v\omega_x - v_x\omega = -\frac{\nu}{\lambda}(-\Delta)^\beta\omega + f$$

by v and integrate on $\mathbb{T}$ to obtain

$$\frac{\nu}{\lambda} \int v(-\Delta)^\beta \omega \, dx = \int f v \, dx \leq \|f\|_{L^2} \|v\|_{L^2}.$$

Because $\omega = (-\Delta)^{\frac{1}{2}} v$, the left-hand side of the aforementioned equals

$$\nu \|(-\Delta)^{\frac{\beta}{2} + \frac{1}{4}} v\|_{L^2}^2.$$

As $\|(-\Delta)^{\frac{\beta}{2} + \frac{1}{4}} v\|_{L^2} \geq \|v\|_{L^2}$, we obtain

$$\|(-\Delta)^{\frac{\beta}{2} - \frac{1}{4}} \omega\|_{L^2} = \|(-\Delta)^{\frac{\beta}{2} + \frac{1}{4}} v\|_{L^2} \leq \frac{C\lambda}{\nu} \|f\|_{L^2}.$$

Again, by lemma 2.1 and the Leray–Schauder theorem, we obtain a stationary solution $\omega \in L^2$. It is a simple matter of bootstrapping (see, for instance, [25, section 9.5]) to show that, actually, $\omega \in H^m$.

Case (iii): large viscosity constant. The proof in this case follows a direct iteration scheme. Recall the form

$$\omega = -\frac{1}{\nu} (-\Delta)^{-\beta} (a \partial_x (v \omega) + (1 + a) v_x \omega - f),$$

and define the sequence $\{\omega^{(n)}\}_{n \geq 1}$ as follows:

$$\omega^{(1)} = \frac{1}{\nu} (-\Delta)^{-\beta} f,$$

$$\omega^{(n+1)} = -\frac{1}{\nu} (-\Delta)^{-\beta} (a \partial_x (v^{(n)} \omega^{(n)}) + (1 + a) v_x^{(n)} \omega^{(n)} - f),$$

$$v^{(n)} = (-\Delta)^{-\frac{1}{2}} \omega^{(n)}.$$

We shall prove that there exists some $\nu^* = \nu^*(a, \beta, f) > 0$, such that if $\nu > \nu^*$, then there is a uniform bound

$$\|\omega^{(n)}\|_{H^1} \leq \frac{2}{\nu} \|f\|_{H^1}, \quad \forall n \geq 1.$$

The proof will be based on induction in n ; note that the estimate is trivial for the base case $n = 1$. We estimate

$$\begin{aligned} \|\omega^{(n+1)}\|_{H^1} &\leq \frac{1}{\nu} \|f\|_{H^1} + \frac{|a|}{\nu} \|(-\Delta)^{-\beta} (\partial_x (v^{(n)} \omega^{(n)}))\|_{H^1} + \frac{|1+a|}{\nu} \|(-\Delta)^{-\beta} (v_x^{(n)} \omega^{(n)})\|_{H^1} \\ &\leq \frac{1}{\nu} \|f\|_{H^1} + \frac{C}{\nu} (|a| + |1+a|) \|\omega^{(n)}\|_{H^1}^2, \end{aligned}$$

where we have used $\beta \geq \frac{1}{2}$, H^1 is an algebra, and $\|v^{(n)}\|_{H^1} \leq C \|\omega^{(n)}\|_{L^2}$. With the induction hypothesis $\|\omega^{(n)}\|_{H^1} \leq \frac{2}{\nu} \|f\|_{H^1}$,

$$\|\omega^{(n+1)}\|_{H^1} \leq \frac{1}{\nu} \|f\|_{H^1} \left(1 + \frac{4C(|a| + |1+a|)}{\nu^2} \|f\|_{H^1} \right) \leq \frac{2}{\nu} \|f\|_{H^1},$$

if $\nu > 4C(|a| + |1 + a|)\|f\|_{H^1} =: \nu^* > 0$. Now, given that we have a uniform H^1 bound, we see that there exists a subsequence that converges strongly in $H^s(\mathbb{T})$ for any $s < 1$ with limit $\omega \in H^1(\mathbb{T})$. It is straightforward to show that ω defines the unique steady solution.

Uniqueness. Assume that there are two H^1 -solutions, ω_1 and ω_2 , to the equation

$$av\omega_x - v_x\omega = -\nu(-\Delta)^\beta\omega + f.$$

We denote the corresponding velocity by v_1 and v_2 . Defining $\bar{\omega} = \omega_1 - \omega_2$ and $\bar{v} = v_1 - v_2$, we obtain (from subtracting and taking $(-\Delta)^{-\frac{1}{2}}$ of both sides)

$$-\nu(-\Delta)^{\beta-\frac{1}{2}}\bar{\omega} = a(-\Delta)^{-\frac{1}{2}}\partial_x(\bar{v}\omega_1 - v_2\bar{\omega}) - (1+a)(-\Delta)^{-\frac{1}{2}}(\bar{v}_x\omega_1 - v_{2,x}\bar{\omega}).$$

Multiplying both sides by $\bar{\omega}$ and integrating in $\mathbb{T}$, we obtain

$$\nu\|(-\Delta)^{\frac{1}{2}(\beta-\frac{1}{2})}\bar{\omega}\|_{L^2}^2 \leq C_a(\|\omega_1\|_{H^1} + \|\omega_2\|_{H^1})\|\bar{\omega}\|_{L^2}^2.$$

Given that $\beta \geq \frac{1}{2}$, we obtain, unless $\bar{\omega} = 0$,

$$\frac{\nu}{C_a} \leq \|\omega_1\|_{H^1} + \|\omega_2\|_{H^1}.$$

Hence, there cannot be two solutions in the ball $\{\omega \in H^1 : \|\omega\|_{H^1} \leq \frac{\nu}{3C_a}\}$. This inequality, for a large enough viscosity, holds for any stationary solution in the cases (i) and (ii) from the *a priori* estimates. The proof is complete. $\square$

Remark 2.3. We numerically show the non-uniqueness of stationary solutions for very large viscosities in the case $\beta = 1, a = \frac{1}{2}$ as follows; see section 3.4 and figure 7. These numerical solutions seem to grow in the order of $O(\nu)$ as $\nu \rightarrow \infty$. (There is always one solution with order $O(\nu^{-1})$).

Remark 2.4. Let us comment on the proof of lemma 2.2. This type of inequality was first obtained in [16] in two dimensions and then was extended in several works [17, 32]. The version that covers the range $p \in (1, 2)$ was obtained only very recently by [7]. Although the proof was given in $\mathbb{R}$, it is straightforward to extend the proof to the case of $\mathbb{T}$, using the representation formula (see [16])

$$(-\Delta)^\beta f(x) = C_\beta \sum_{n \in \mathbb{Z}} \text{P.V.} \int_{\mathbb{T}} \frac{f(x) - f(y)}{|x - y - 2\pi n|^{1+2\beta}} dy.$$

2.2. Existence and nonexistence in the case $a = 0$

By directly using the series expansion, we can prove existence of a unique stationary solution in the special case $a = 0$ when the viscosity constant is not too small. More interestingly, we can prove the *nonexistence* of a stationary solution if the viscosity constant is smaller than a critical threshold. In this case, we fix the forcing $f = \frac{1}{2} \sin 2x$ for simplicity, although the same analysis is expected to carry over to the case of general odd and smooth forcing. The stationary equation then becomes

$$-2\omega H[\omega] = -2\nu(-\Delta)^\beta\omega + \sin 2x,$$

and a solution ω to the aforementioned is forced to have only even sine coefficients:

$$\omega(x) = \sum_{n \geq 1} b(2n) \sin(2nx).$$

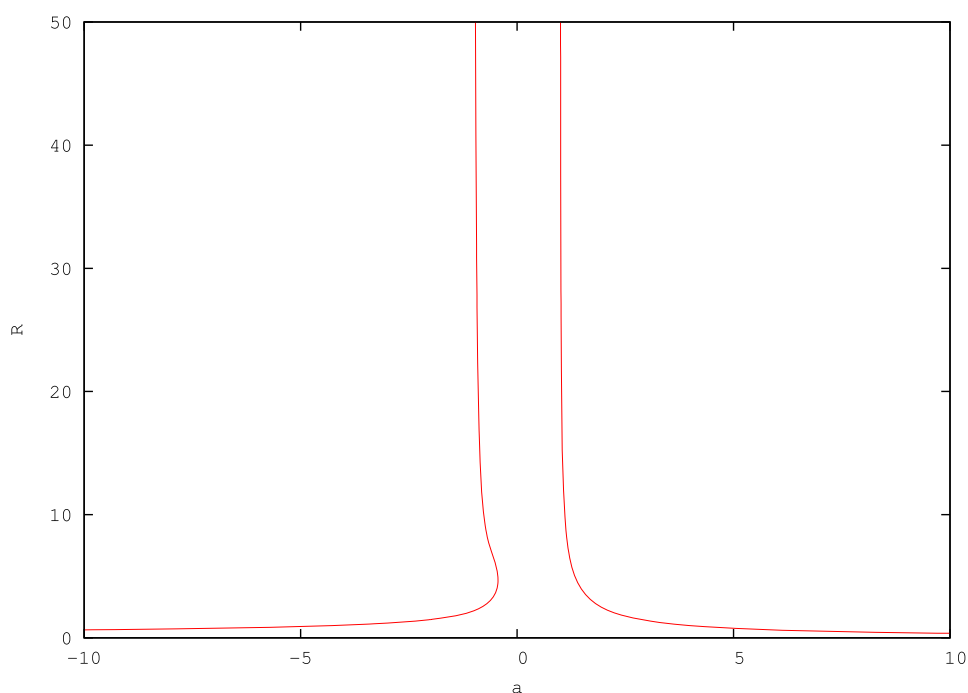


Figure 1. Bifurcation curves for $-10 \leq a \leq 10$ if $\beta = 1$

Thus, it follows that

$$\sum_{k=1}^{n-1} b(2k)b(2(n-k)) = -2(2n)^{2\beta}\nu b(2n), \quad n \geq 2,$$

and

$$0 = -2^{1+2\beta}\nu b(2) + 1.$$

Now, we recall the following lemma:

Lemma 2.5 ([50, lemma 6]). *Given $\alpha > 1$, define a sequence recursively as*

$$p_1^{(\alpha)} = 1, \quad p_n^{(\alpha)} = \frac{1}{n^\alpha} \sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)}, \quad n \geq 2.$$

Thus,

$$D_\alpha n^{\alpha-1} e^{-3\alpha n} < p_n^{(\alpha)}$$

where D_α is a constant satisfying $\alpha 2^{3\alpha} < D_\alpha \leq e^{3\alpha}$.

This lemma also holds in the case $\alpha = 1$, with a simpler proof. To apply this lemma, we define $c_n = -\frac{1}{2^{1+2\beta}\nu} b(2n)$ to obtain

$$\frac{1}{n^{2\beta}} \sum_{k=1}^{n-1} c_k c_{n-k} = c_n, \quad n \geq 2, \quad c_1 = -\frac{1}{(2^{1+2\beta}\nu)^2}.$$

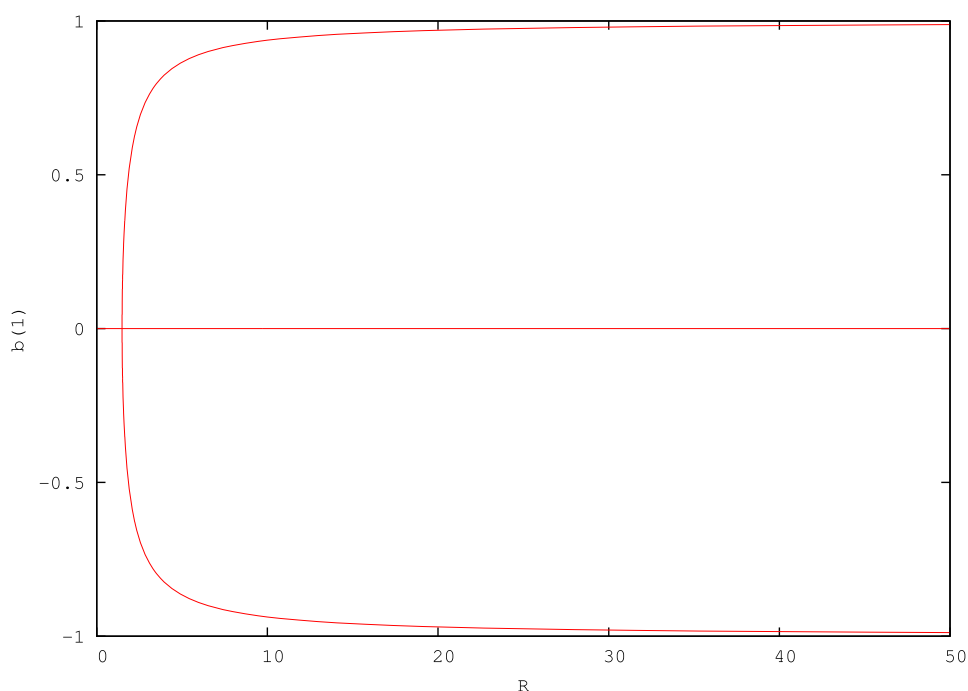


Figure 2. Bifurcation diagram for $a = -2$.

Hence, we have

$$p_n^{(2\beta)} = -(2^{1+2\beta}\nu)^{2n-1}b(2n),$$

which results in (after the application of lemma 2.5)

$$|b(2n)| \geq C_\beta n^{2\beta-1} \frac{e^{-6\beta n}}{(2^{1+2\beta}\nu)^{2n-1}}$$

for some constant $C_\beta > 0$ depending only on $\beta > \frac{1}{2}$. It is easy to see from the recursion for $b(2n)$ that for fixed β and n , $|b(2n)|$ increases as $\nu \rightarrow 0$. Moreover, for a large enough $\nu > 0$, it is easy to see that $|b(2n)|$ decays exponentially in n . We have arrived at the following proposition.

Proposition 2.6. Consider (2.1) with $a = 0$ and $f = \frac{1}{2} \sin 2x$. Thus, for any $\beta \geq \frac{1}{2}$, there exists $C_\beta > 0$ such that the following holds:

- for $\nu > C_\beta$, there exists a unique L^2 solution to (2.1),
- for $0 < \nu \leq C_\beta$, there is no L^2 solution to (2.1).

Remark 2.7. Let us give a few simple remarks related to proposition 2.6.

- Note that at the critical viscosity $\nu = C_\beta$, existence cannot occur because, otherwise, we can extend the solution to slightly larger $\nu > 0$ by continuity. More specifically, we have a rough estimate $C_1 \approx 1/15.5 = 0.645 \dots$, $C_{1/2} \approx 1/20.96 \dots = 0.0477 \dots$ from numerical computation. (See a later section for details.)

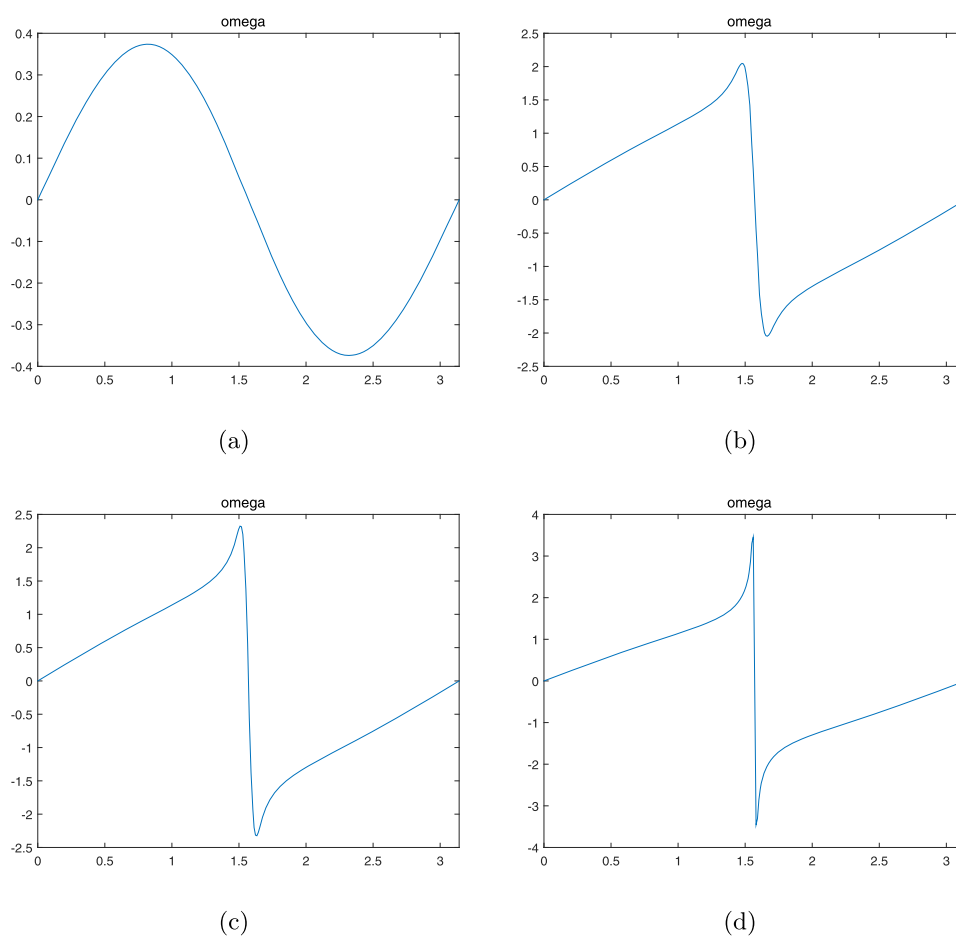


Figure 3. Basic solutions for $a = -2$ and various R : (a) $R = 1$, (b) $R = 50$, (c) $R = 100$, (d) $R = 1000$.

- It is not difficult to show that there cannot exist nontrivial time-periodic solutions, for any values of $\nu \geq 0$ and β . To see this, one can expand $\omega(t, x) = \sum_{n \geq 1} b(t, n) \sin(nx)$ and then the equations for $b(t, 1)$ and $b(t, 2)$ read

$$\begin{aligned} \frac{d}{dt} b(t, 1) &= -\nu b(t, 1), \\ \frac{d}{dt} b(t, 2) &= \frac{1}{2} - \frac{1}{2} b^2(t, 1) - 2^{2\beta} \nu b(t, 2). \end{aligned}$$

Unfortunately, this ODE system does not admit any time-periodic solution, apart from the fixed point $b(t, 1) = 0, b(t, 2) = (2^{1+2\beta} \nu)^{-1}$.

3. Bifurcations and inviscid limits for various a and β

In the previous sections, we have studied the results of testing the uniqueness and existence of a stationary solution for various a and β . In general, we expect uniqueness and existence for

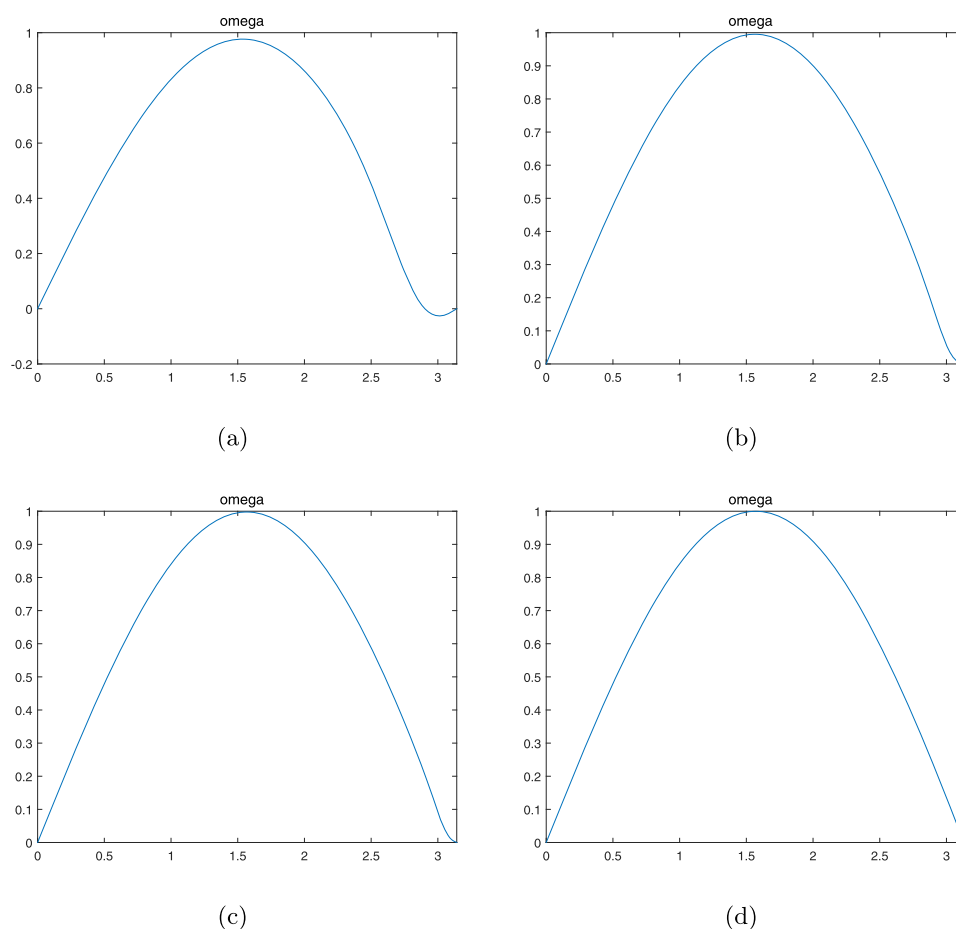


Figure 4. Bifurcated solutions for $a = -2$ and various R : (a) $R = 1$, (b) $R = 50$, (c) $R = 100$, (d) $R = 1000$.

some large ν (or for a sufficiently small R). Therefore, it will be interesting to see what happens with a unique solution if we decrease ν or equivalently increase R . Here, we investigate this problem via proper numerical computation and asymptotic analysis. Specifically, we solve (1.8) using the path continuation method [27]. We use R (instead of ν) for the convenience of the current discussion. The present numerical approach has been successful for the second author for obtaining interesting results for similar equations under different scales of forcing. See [34–37].

Let us shortly explain the numerical method used in this study. We consider the generalized CLM equation with viscosity order $\beta > 0$, as in (1.8). We assign

$$\omega^N = \sum_{n=1}^N b(n) \sin nx, \quad v^N = -\sum_{n=1}^N \frac{b(n)}{n} \sin nx \quad (3.1)$$

as approximations that are substituted into (1.8). (The constant term of v is set to zero.) By considering the coefficients of $\sin kx$ ($k = 1, 2, \dots, N$) in the resulting equation from (1.8), we may derive a system of N algebraic equations on $b(1), \dots, b(N)$. We begin by finding the

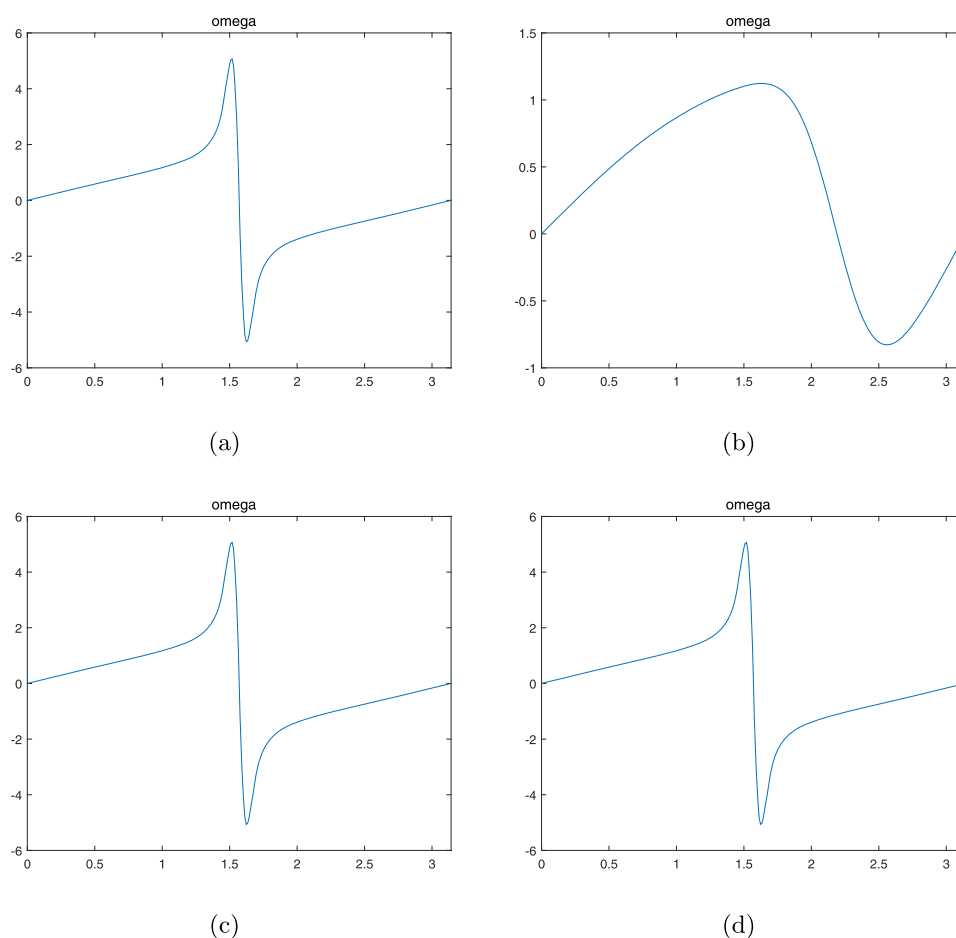


Figure 5. Various solutions for $a = -1/2$: (a) basic solution at $R = 100$, (b) bifurcated solution at $R = 9$, (c), (d) bifurcated solutions from two different branches at $R = 100$, but they are almost equal to the basic solution.

unique solution (we will call this *the basic solution*) for $\nu = R = 1$ (or even smaller $R < 1$, if necessary) via Newton iteration, from which we continue $b(1), \dots, b(N)$ using the numerical path continuation method as R increases. Specifically, we adopt the numerical package AUTO [1], regarding which more details are described in [35].

In a test for convergence, we employ the following criterion:

$$\frac{\|\omega^N - \omega^{2N}\|}{\|\omega^N\|} < 10^{-3}, \quad (3.2)$$

where $\|\cdot\|$ is the L^2 -norm, and ω^N denotes the approximate solution in (3.1) with mode N . We increase N until this criterion is satisfied. For most of the computations in this paper, we adopt $N = 1000$, which is checked to satisfy (3.2) up to $R = 50\,000$, the upper limit for the current investigation. We also compute with different numbers of Fourier modes, such as $N = 500, 1500, 2000$, and even $N = 5000$ for the confirmation of the numerical results.

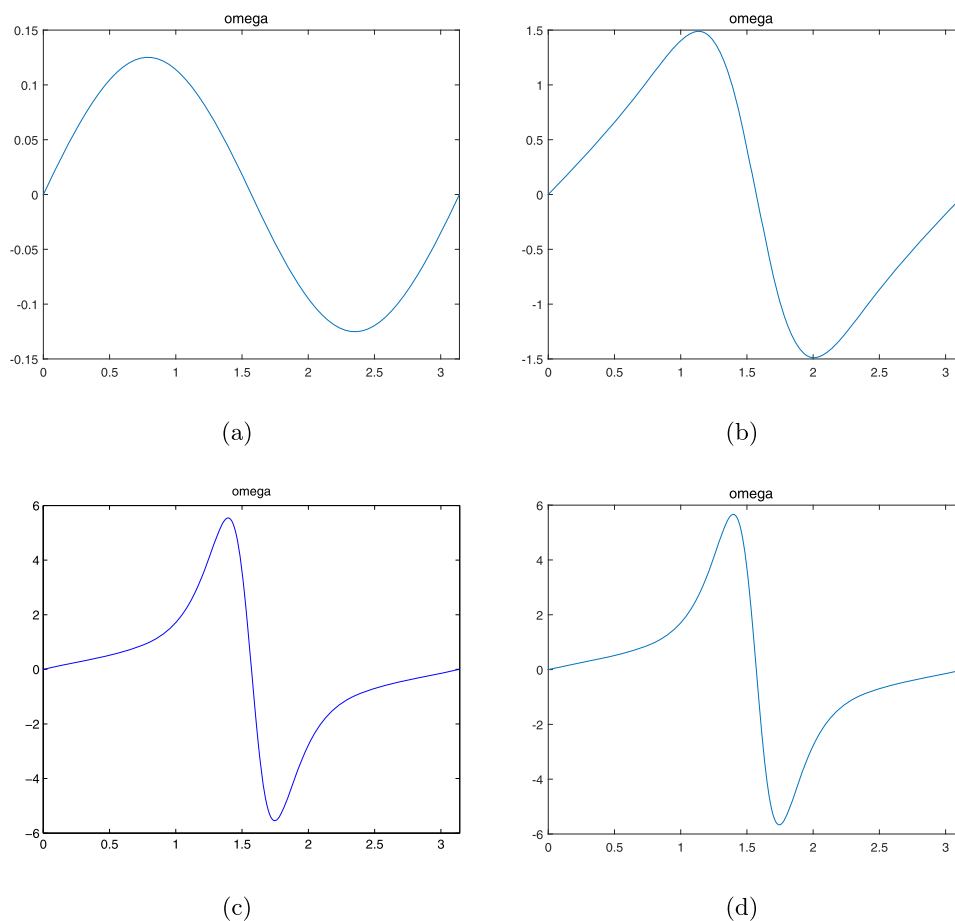


Figure 6. Basic solutions for $a = 0$ and various R : (a) $R = 1$, (b) $R = 10$, (c) $R = 15$, (d) $R = 15.5$.

3.1. Bifurcation curves for various values of a

We start the discussion with fixing $\beta = 1$ for the following sections. We shall show the corresponding results for $\beta = 1/2$ in another separate section later. The numerical results reveal that for various values of a , one or two (or even more) bifurcations occur with (or without) Hopf bifurcations. The bifurcations are found to be mostly supercritical, whereas some of them are determined to be turning points. Generally speaking, in the supercritical bifurcation case, if there appears a bifurcation for certain values of a , two new solutions are generated, and they often approach $\pm \sin x$ as $R \rightarrow \infty$, for which we expect the inviscid limit solution. However, for some cases (see further sections for more details), new emerging solutions do not approach the supposed inviscid limit. Overall, we found more complicated results than those from the gPJ equation [29].

As the general bifurcation diagram is considerably complicated and currently not a major concern, we focus on the first bifurcation point for increasing or decreasing Reynolds number from $R = 1$. We present the numerical results for various values of a in figure 1. For the range $-10 \leq a \leq 0$, we clearly observe a turning point at $a = -1/2$, where the curve goes back and

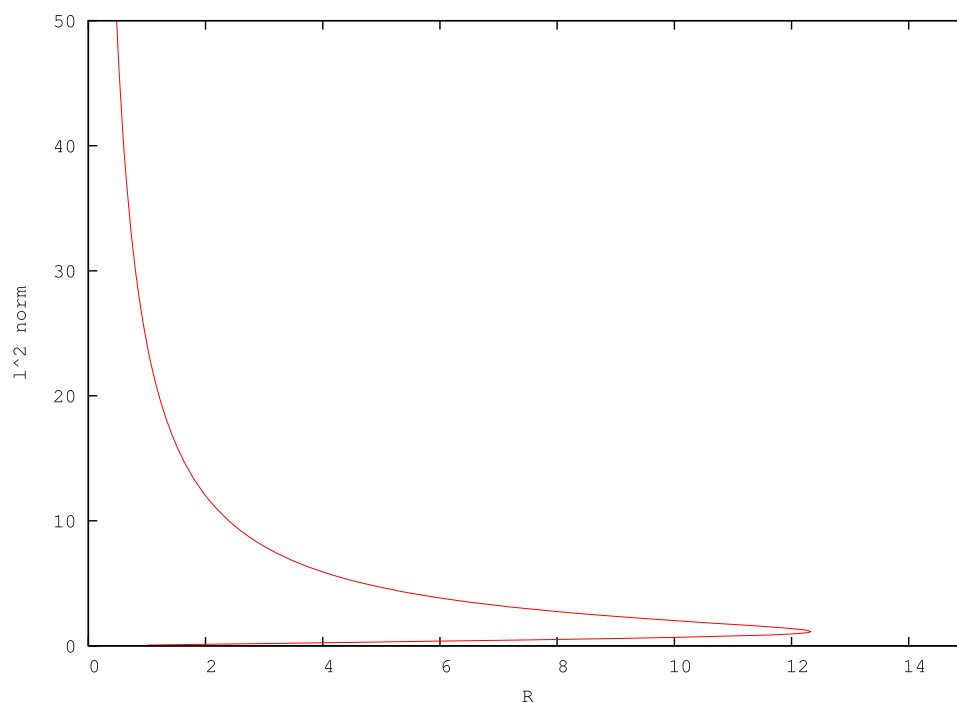


Figure 7. Bifurcation diagram for $a = 1/2$.

increases rapidly indefinitely near $a = -1$ as $R \rightarrow \infty$. Therefore, more than two bifurcations clearly exist for $-1 < a < -1/2$.

On the other hand, for the range $0 \leq a \leq 10$, we do not see any turning point in the diagram. In fact, the bifurcation point goes to infinity as $a \rightarrow 1$ from the right, and no bifurcation appears for $0 < a < 1$. (The case $a = 1$ is trivial because the forcing vanishes, and the zero solution produces no bifurcation for any R .) From these results, we numerically confirm that no bifurcation happens for $-1/2 < a \leq 1$.

3.2. Case of $-10 \leq a < -1/2$

For this range of a , we generally face two bifurcations: one near $R = 0$ (much less than $R = 1$, which is not our concern in this study), and the other one for $R > 1$. In general, the (first) bifurcation of $R > 1$ occurs later, i.e., for a larger R , as a increases in this range. For example, if $a = -2$, the bifurcations occur at $R \approx \{0.097, 1.484\}$. The bifurcation diagram for $a = -2$ is shown in figure 2, from which we check that the new bifurcated solutions clearly approach $\pm \sin x$ as inviscid limits.

This trend changes at $a = -1$, where we have a fold point at $R \approx 0.010$ and a bifurcation point at $R \approx 2.693$. Now, for all a in this range, the basic solution does not approach the supposed inviscid limit, as shown in figure 3. In fact, the solution becomes steep and eventually blows up as R increases. We analyse this singular behaviour later in section 3.7.

The bifurcated solutions at various R are then shown in figure 4 with the corresponding proposed inviscid limits.

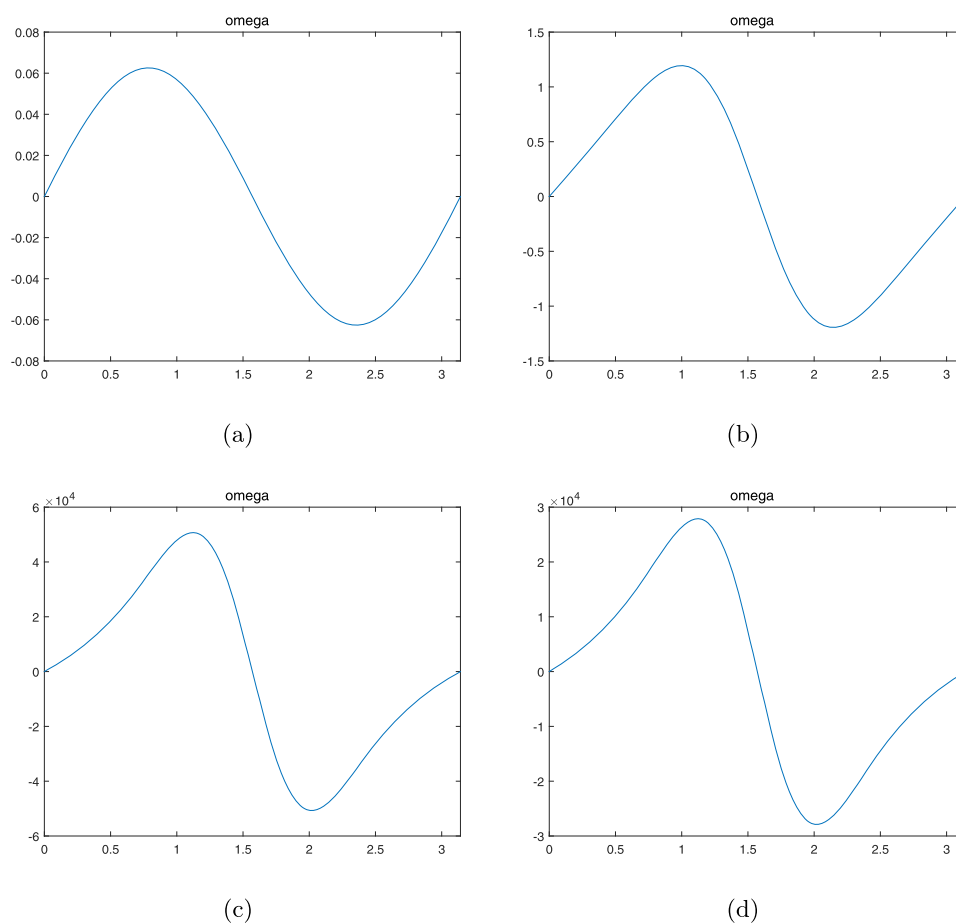


Figure 8. Basic solutions for $a = 1/2$ and various R : (a) $R = 1$, (b) $R = 12.325$, (c) $R = 0.00543$, (d) $R = 0.0009$.

3.3. Case of $a = -1/2$

One interesting case is when $a = -1/2$. Although a few bifurcations appear at $R \approx \{5.175, 8.475, 8650, \dots\}$, and new corresponding solutions emerge (we observe that some of them are identical solutions), the inviscid limit of these are not $\pm \sin x$, as shown in figure 5. Moreover, we note that the solution in (b) is not symmetric w.r.t. $x = \pi/2$, which is interesting in the sense that almost all numerical solutions we have obtained are symmetric. We do not know the reason for this peculiarity. (We also comment that this case has some connection to geodesics on a certain orientation-preserving group in geometry [48]).

Thus, there is no limiting solution approaching $\pm \sin x$ for this range of a .

3.4. Case of $-1/2 < a \leq 1$

In this range of a , the situation changes again. No bifurcation appears for any R , and the limiting solution is not what we expected, i.e., $\pm \sin x$, in figure 8. The case $a = 0$ is exemplified as in figure 6, where the solution develops a singularity near $x = \pi/2$ as R increases. Thus, no bifurcations appear for $a = 0$, which corresponds to proposition 2.6, which is explained

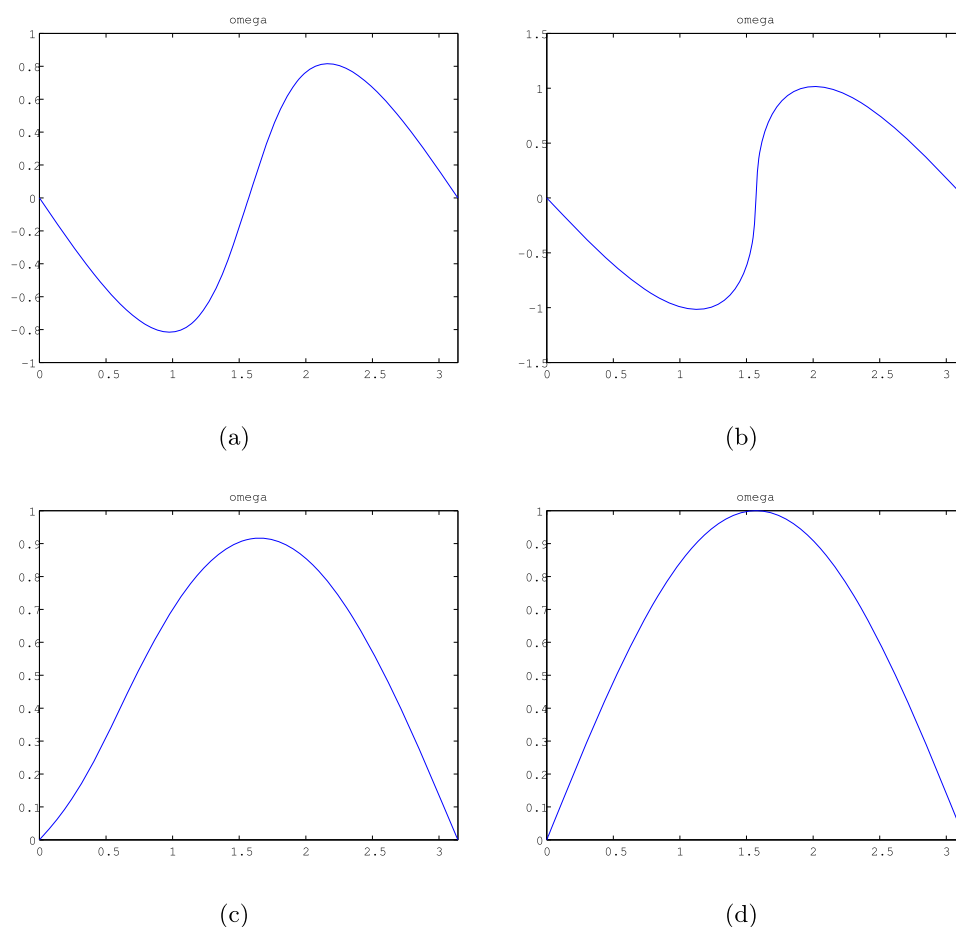


Figure 9. Basic solutions (a), (b) and bifurcated solutions (c), (d) for various R for $a = 2$: (a), (c) $R = 10$; (b), (d) $R = 5000$.

in a previous section. In fact, the numerical solution blows up at around $R \approx 15.5$ in the l^2 norm.

The case $a = 1/2$ is also interesting in the sense that a fold point appears ($R \approx 12.325$) and that the basic solution branch turns back and goes backward (with decreasing R) and eventually blows up near $R = 0$. See figure 7 for the bifurcation diagram and figure 8 for the solutions.

3.5. Case of $1 < a \leq 10$

A bifurcation occurs, and new solutions have been generated. For the case of $a = 2$, a supercritical bifurcation happens at around $R = 2.3997 \dots$, and a newly generated solution approaches $\pm \sin x$ as $R \rightarrow \infty$. Basic solutions and bifurcated solutions are shown for both $R = 10$ and 5000 in figure 9. Interestingly, for larger values, e.g., $a = 5, 10$, bifurcations occur early, with the bifurcated solutions approaching the supposed inviscid limit $\pm \sin x$.

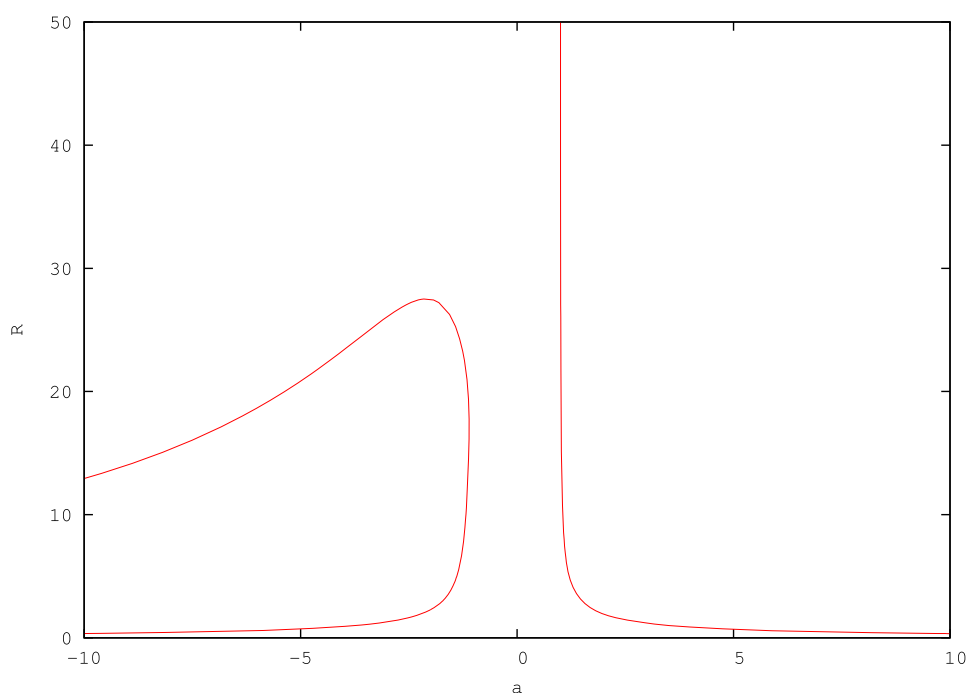


Figure 10. Bifurcation curves for $-10 \leq a \leq 10$ if $\beta = 1/2$.

3.6. $\beta = 1/2$

We also compute a case of lower dissipation with $\beta = 1/2$ for various a and compare the results with those of the previous corresponding cases for $\beta = 1$. Broadly speaking, the numerical results exhibit almost no qualitative difference for the two cases with different dissipation terms, except for the $a = -1$ case. On the other hand, we find a general quantitative change in that the bifurcation appears somewhat later for bigger R than for the corresponding $\beta = 1$ case. Furthermore, if the solution blows up in the $\beta = 1/2$ case, it happens for the smaller R . We present the bifurcation curves in figure 10.

The only exceptional case from this general rule is when $a = -1$. With $\beta = 1/2$, we observe a supercritical bifurcation at $R \approx 20.79$ similar to in the previous case, but the bifurcated solution does not approach the inviscid limit. The new stable branch turns back, and R decreases up to $R \approx 0.1624$. Therefore, we suppose a multiplicity of solutions for the rather small $R > 0.1624$ from the numerical results; see figure 11. On the other hand, the basic solution continues for large R but does not approach the inviscid limit; see figure 12. (In contrast to this, if $\beta = 1$, the bifurcated solution approaches the inviscid limit.) We do not know any analytical explanation for this phenomenon with the smaller dissipation order.

We then summarize in table 1 the fundamental results concerning the existence of bifurcation, and the convergence to the supposed inviscid limit for various values of a . (This is for the case of $\beta = 1$, but for $\beta = 1/2$, the situation is almost same except for $a = -1$, as already reported).

3.7. Inviscid limit singular solutions

From the extensive numerical experiments, we observe a common interesting feature of the singular limit solution as $R \rightarrow \infty$. As stated previously, for some a, β , bifurcations appear,

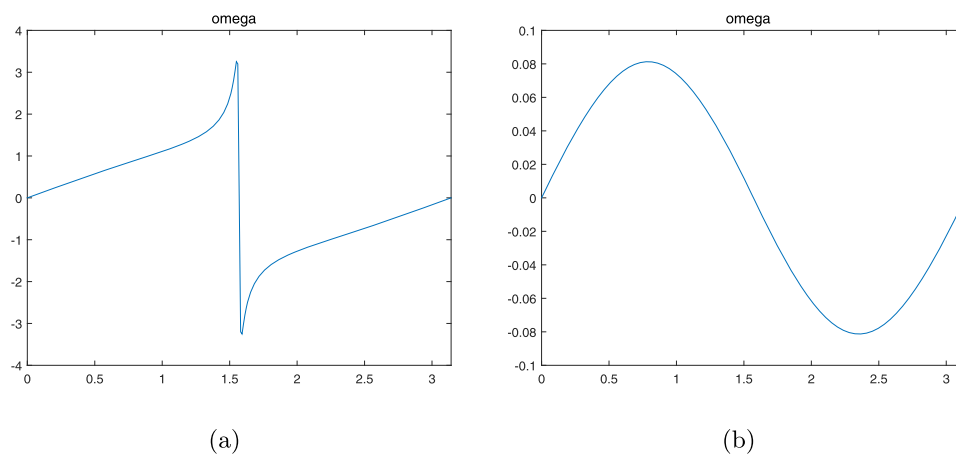


Figure 11. Bifurcated solutions for $a = -1$ and various R : (a) $R = 10$, (b) $R = 0.1624$.

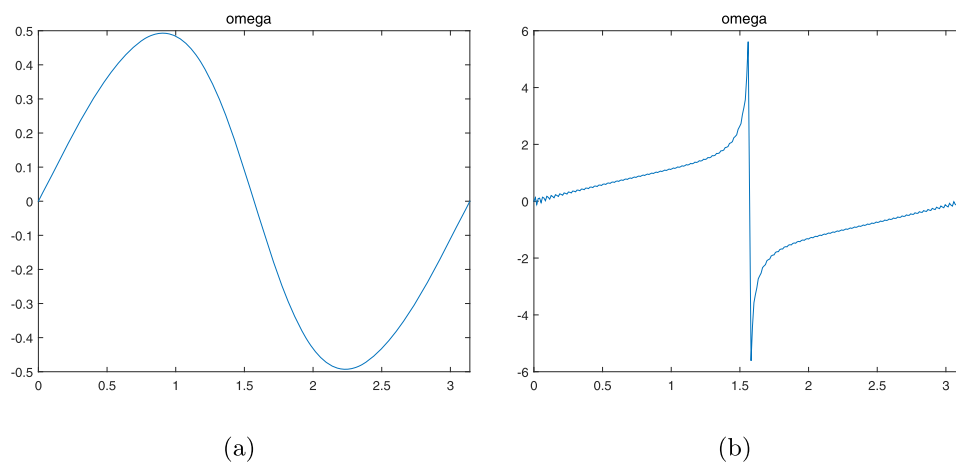


Figure 12. Basic solutions for $a = -1$ and various R : (a) $R = 1$, (b) $R = 20$.

Table 1. Summary of inviscid limit phenomena for various a (with fixed $\beta = 1$).

Range of a	$-10 \leq a < -1/2$	$a = -1/2$	$-1/2 < a \leq 1$	$1 < a \leq 10$
Bifurcation	o	o	x	o
Inviscid limit (basic sol.)	x	x	x	x
Inviscid limit (bifurcated sol.)	o	x	x	o

with the corresponding limit solution as $\pm \sin x$, as expected. On the other hand, in the same limit, the basic solution eventually becomes singular with a very sharp changing interior layer near $x = \pi/2$. In fact, the specific asymptotic profile of ω changes according to the values of a and possibly β , as seen in figure 13, for certain specific examples. (For a brief discussion, we now fix $\beta = 1$).

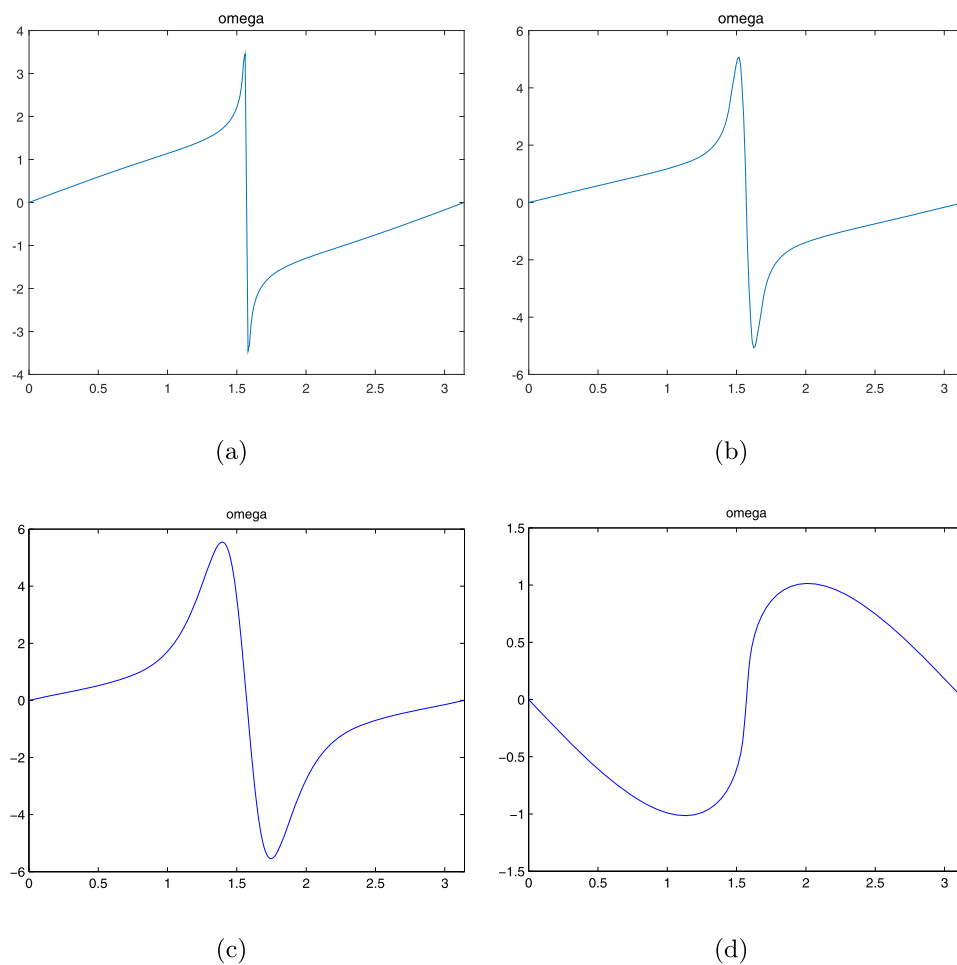


Figure 13. Basic solutions for various a : (a) $a = -2, R = 1000$, (b) $a = -1/2, R = 100$, (c) $a = 0, R = 15$, (d) $a = 2, R = 1000$.

Among these possible limits, we observe a common type, which consists of mostly (almost) a linear part attached to a very thin singular layer near $x = \pi/2$. For instance, if $a = -2$, the basic solution exhibits the distinct limit of a linear solution outside the interior layer (near $x = \pi/2$); this solution is computable for a rather large $R = 10,000$ or higher with enough number of modes. On the other hand, for $a = 0$, the basic solution exhibits no specific limit function there, and, in our computation, the numerical solution blows up for a rather small $R \approx 20$. For $a = 2$, the basic solution again exhibits no distinct limit in most of the region but does not blow up until $R = 10,000$. In this respect, we conclude that a variety of possibilities may exist for the inviscid limits of the basic solutions for different a values.

For almost all cases, it is very difficult to determine the specific limit expression (exactly or asymptotically) of the solution from numerical results with reasonable ansatz. In fact, the problem is harder than the gPJ equation case [29] because the corresponding equation now contains two unknown functions ω, v interrelated by the Hilbert transform. In contrast, the generalized PJ equation is represented in terms of a single unknown function.

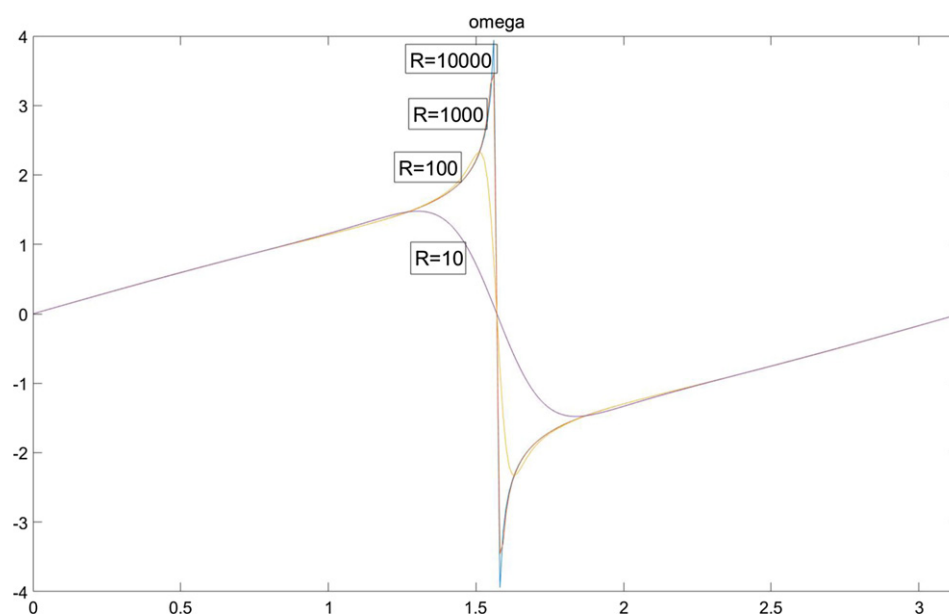


Figure 14. Basic solutions for $a = -2$ and various $R = \{10, 100, 1000, 10\,000\}$ with $N = 5000$.

Nonetheless, we here provide certain partial information on the singular limit, namely the order of the singularity at $x = \pi/2$, and the slope of the function at $x = 0$. Note that in the special case $a = -2$, the limit function for ω , at least, seems to be linear near the origin, as shown in figure 13. Clearly, the numerical result shows that the limit solution is composed of two different regions. First, the outer region corresponds to most of the whole domain, except for a thin layer near $x = \pi/2$, which will be called the inner region or interior layer with abruptly changing vorticity. See figure 14.

3.7.1. Slope at $x = 0$. In the outer region, as $R \rightarrow \infty$, numerical data suggest that the vorticity ω approaches a linear function passing the origin at least near $(0, 0)$. Let us express this and the corresponding v as

$$\omega = \omega_0 x + \cdots \quad (3.3)$$

$$v = v_0 x + \cdots \quad (3.4)$$

Inserting these into the inviscid limit equation

$$av\omega_x - v_x\omega = \frac{1-a}{2} \sin(2x), \quad (3.5)$$

and comparing the leading order terms, we have the condition

$$\omega_0 v_0 = -1, \quad v_0 = -1/\omega_0. \quad (3.6)$$

Now, it is important to note that although ω is almost linear in most of the outer region, v is not (figure 15). This makes it more difficult to determine the inviscid limit solution. However,

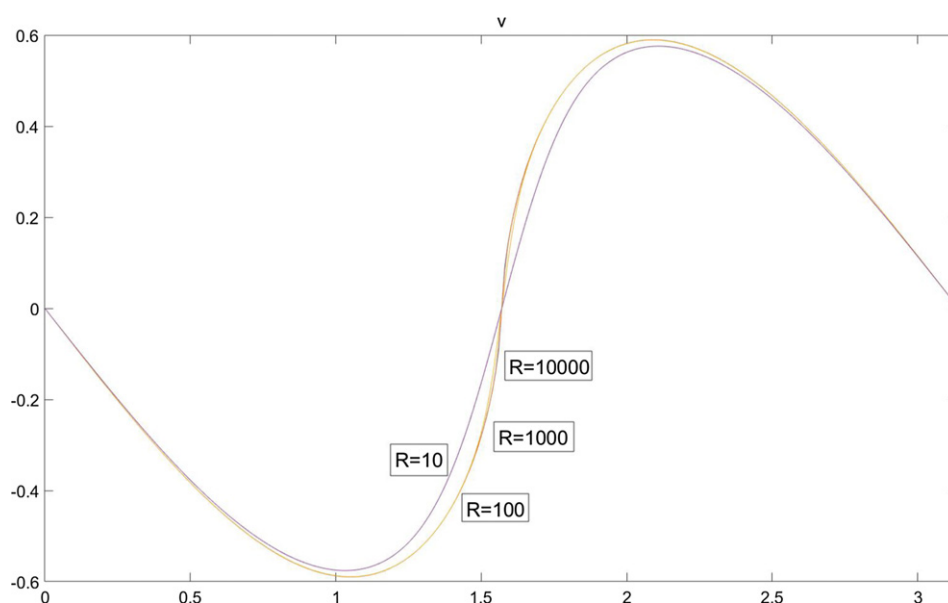


Figure 15. Velocity v for $a = -2$ and various $R = \{10, 100, 1000, 10\,000\}$ with $N = 5000$.

we can compute an approximate value of ω_0 , which is valid for any a in the following heuristic argument. From

$$v'(0) = v_0 = -\frac{1}{\omega_0} = H[\omega](0),$$

we estimate

$$H[\omega](0) = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega(x)}{0-x} dx \approx -\frac{\omega_0}{\pi} \sum_{n=-\infty}^{\infty} \int_{(n-1/2)\pi}^{(n+1/2)\pi} \frac{x-n\pi}{x} dx \quad (3.7)$$

by considering the leading order term only. In the previous expression, rewriting with $x - n\pi = y$ and interchanging the order of summation and integration, we calculate

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{y}{y+n\pi} dy = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} y \cot y dy = \log 2.$$

Subsequently, equating the two representations of $v_x(0) = \lim_{x \rightarrow 0} v_x$ given in (3.6) and in (3.7) as

$$-1/\omega_0 \approx -\omega_0 \log 2$$

produces the approximate evaluation of the slope

$$\omega_0 \approx \pm (\log 2)^{-1/2} = \pm 1.201 \dots$$

This agrees quite well with the numerical data near $(0, 0)$, as in figures 16 and 17. Notably, the value of ω_0 is independent of a , and only its sign changes. Specifically, the numerical results

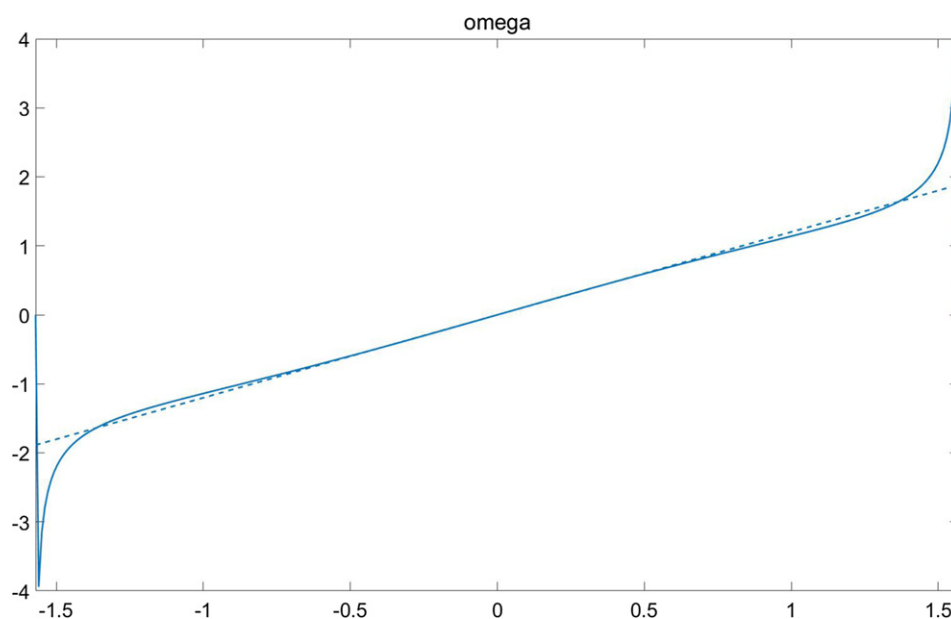


Figure 16. Basic solution for $R = 10\,000$ with $N = 5000$ for $a = -2$ (solid line) and the approximate line with slope 1.201 (dashed line).

demonstrate that the sign of the slope is determined according to whether $a < 1$ (positive slope) or $a > 1$ (negative slope). We have no rigorous proof of this phenomenon, but let us suggest a reasonable guess that it is related to the profile of the solution for very small $R > 0$ (or very large $\nu \gg 1$). In that case, the basic solution is approximated by the viscous term with the outer forcing in (2.1); that is,

$$-\frac{1}{R}(-\Delta)^\beta \omega + \frac{1-a}{2} \sin 2x \approx 0. \quad (3.8)$$

However, in this equation, the leading order term of ω near $x = 0$ is determined by the signature of $1 - a$; that is, positive if $a < 1$, or negative if $a > 1$. Thus, it is reasonable that the basic solution continues in such a way that there is no critical change as R increases, such that the inviscid limit solution maintains the sign of the slope.

3.72. Order of singularity at $x = \pi/2$. In addition, in the interior layer near the singular point $x = \pi/2$, we can calculate the solution asymptotically as follows. Assuming a singularity at $x = \pi/2$, we suppose the solution

$$\omega \sim A \left(\frac{\pi}{2} - x \right)^{-\gamma}, \quad A, \gamma > 0.$$

Furthermore, v has the order of singularity $1 - \gamma > 0$ at the same point from the property of Hilbert transform (see lemma A.1 in the appendix A) and the definition of v , and thus we assign

$$v \sim A' \left(\frac{\pi}{2} - x \right)^{1-\gamma}, \quad A' > 0.$$

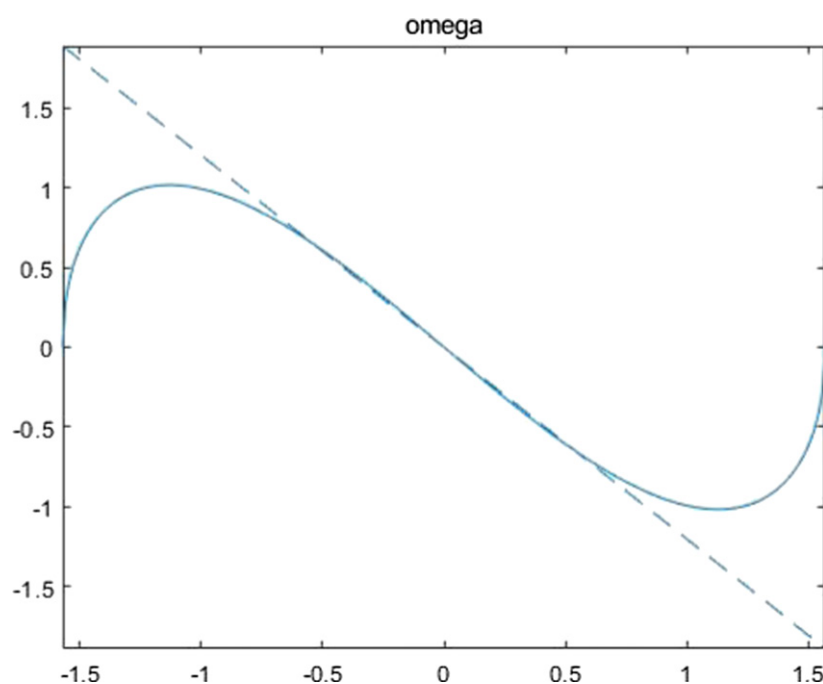


Figure 17. Basic solution for $R = 10\,000$ with $N = 5000$ for $a = 2$ (solid line) and the approximate line with slope -1.201 (dashed line).

Inserting these into (3.5) and considering leading order terms, we obtain the relation

$$(1 - \gamma) + a\gamma = 0, \quad \gamma = \frac{1}{1-a}.$$

For this to be a reasonable ansatz for ω , we need at least $-\gamma > -1$ for the Hilbert transform to be well-defined. Hence, we predict that for $a < 0$, $\gamma = \frac{1}{1-a}$. In particular, when $a = -2$, we obtain $\gamma = \frac{1}{3}$. This is in good agreement with the numerical results in figure 18.

On the other hand, for $a > 0$, we suppose that

$$\omega \sim A \left(\frac{\pi}{2} - x \right)^\eta, \quad A, \eta > 0. \quad (3.9)$$

In this case, $H[\omega] = \partial_x v$ attains a finite value at $x = \pi/2$ (which follows from lemma A.1), and therefore

$$v \sim A' \left(\frac{\pi}{2} - x \right).$$

Inserting these approximations for ω and v into (3.5), we obtain

$$a\eta - 1 = 0, \quad \eta = \frac{1}{a}.$$

In the preceding argument, we have implicitly assumed that $|av\omega_x - v_x\omega| \gtrsim |\sin 2x|$ holds for $x \sim \pi/2$, which requires $\eta < 1$. Hence, we are able to predict that, when $a > 1$, $\eta = \frac{1}{a}$. However, numerical data reveal that it is a rough estimate. For instance, in the case of $a = 2$, the

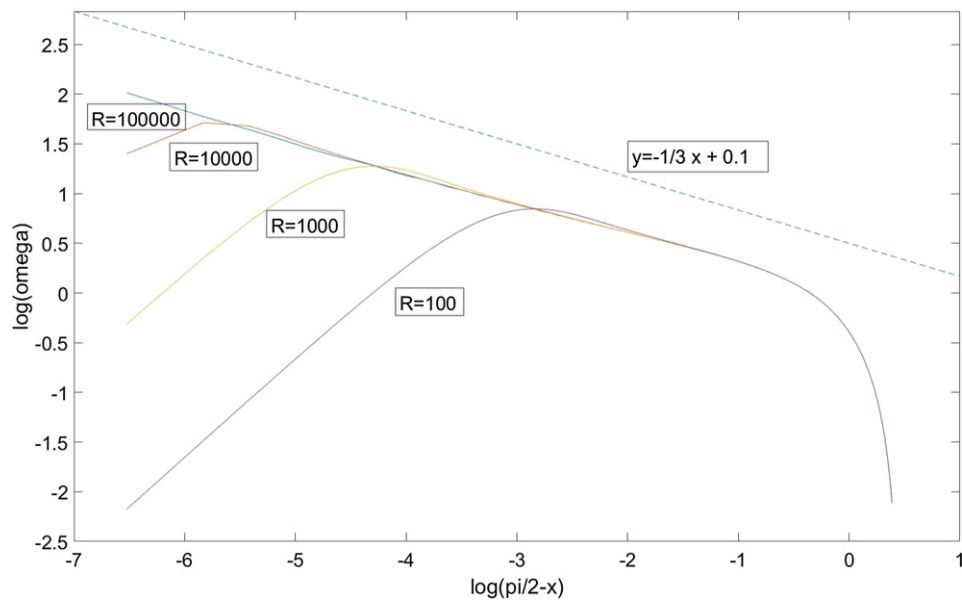


Figure 18. Loglog plot of ω for $R = \{100, 1000, 10\,000, 100\,000\}$ with $N = 5000$ for $a = -2$ and the approximate line with slope $-1/3$ (dashed line).

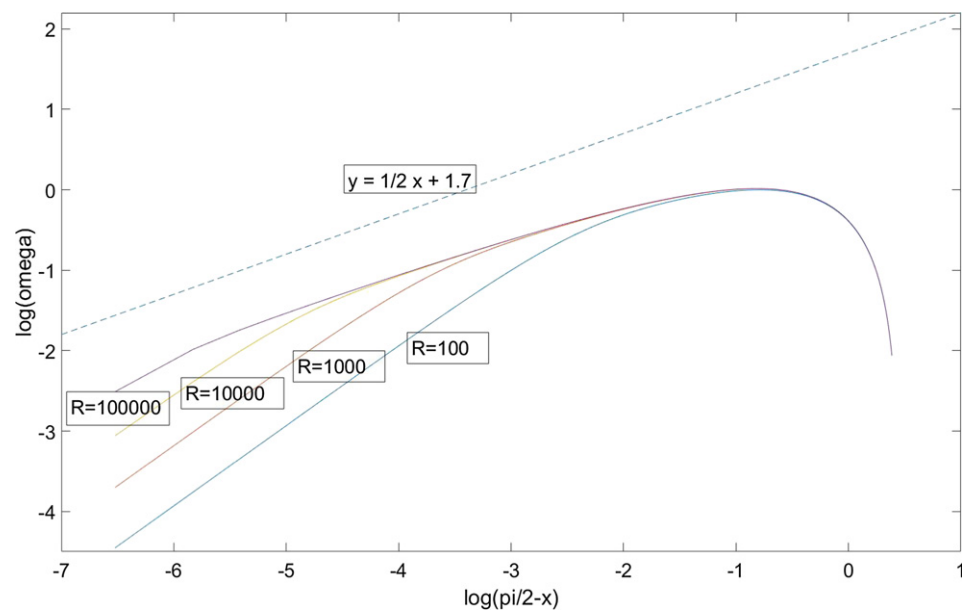


Figure 19. Loglog plot of ω for $R = \{100, 1000, 10\,000, 100\,000\}$ with $N = 5000$ for $a = 2$ and the approximate line with slope $1/2$ (dashed line).

slope seems to deviate somewhat from $1/2$, as seen in figure 19. As the previously stated asymptotic argument seems legitimate overall, we suspect that the starting assumption (3.9)

might be significantly rough. Therefore, for a better result, we may need to consider a more complicated asymptotic form of ω than (3.9), possibly containing logarithmic corrections.

4. Conclusion

In this study, we investigated the generalized CLM equation with viscosity and a forcing term. We assigned external forcing, which is a constant multiple of $\sin 2x$ (independent of the viscosity), for $\sin x$, in the inviscid case, to become steady-state. Using Galerkin approximation and path continuation techniques, we numerically computed the branch of stable steady states in the vanishing viscosity limit $\nu \rightarrow 0$. Whereas a variety of interesting phenomena occur in the inviscid limit, we confirmed numerically that when the strength of dissipation is given by $\beta = 1$ and $1/2$, the stable steady states indeed converge to $\sin x$ for values of a outside $[-1/2, 1]$. In contrast, we observed that the basic steady state becomes singular as the viscosity tends to zero, and computed theoretically the order of the singularity and almost linear slope near the origin using properties of the Hilbert transform.

Although the generalized CLM equation may seem artificial, they not only formally model 2D and 3D fluid systems, but can also be derived from higher-dimensional fluid equations upon consideration of solutions with special forms. In the latter sense, these equations are comparable to the gPJ equation. Our theoretical and numerical results again confirmed that these model equations share some features with the 2D Navier–Stokes equations, which have been noted in the works of Matsumoto and Sakajo [44, 45]. Although these equations are one-dimensional, they exhibit very complicated bifurcation and inviscid limit phenomena. Unfortunately, the current paper is mainly focused on finding the branch of solutions converging to the intended limit, and lacks rigorous justification of convergence. We hope that further numerical and theoretical study would help us in getting more insight into the corresponding problems for the actual fluid equations.

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Appendix A. Some properties of Hilbert transform

We collect a few results relevant to the Hilbert transform. The following lemma describes the behaviour of the Hilbert transform acting on functions of the form $g(x) \sim \operatorname{sgn}(x)|x|^\alpha$ near $x = 0$. The result reveals that $H[g](x) \sim c_0 + c_\alpha|x|^\alpha$ near $x = 0$. This was proved in [22] for functions defined on $\mathbb{R}$, but the proof can be easily adapted to the case of $\mathbb{T}$.

Lemma A.1 (See [22, lemma 2.2]). *Assume that $g(x) = \operatorname{sgn}(x)F(|x|^\alpha)$ for some $0 < \alpha < 1$*

where F is a smooth and decaying function satisfying $F(0) = 0$. Thus,

$$\lim_{x \rightarrow 0} \frac{H[g](x) - H[g](0)}{|x|^\alpha} = \cot\left(\frac{\alpha\pi}{2}\right) F'(0).$$

We now extend the aforementioned to the case where $-1 < \alpha < 0$. Formally, using L'Hôpital's rule to the aforementioned, we have

$$\cot\left(\frac{\alpha\pi}{2}\right) F'(0) = \lim_{x \rightarrow 0} \frac{\partial_x(H[g](x) - H[g](0))}{\partial_x(|x|^\alpha)} = \lim_{x \rightarrow 0} \frac{H[\partial_x g](x)}{\alpha \operatorname{sgn}(x)|x|^{\alpha-1}},$$

where we have used the differentiation commutes with taking the Hilbert transform. Because

$$\partial_x g(x) = \alpha|x|^{\alpha-1}F'(|x|^\alpha),$$

the previous computation suggests that the Hilbert transform of a function f with $f(x) \sim |x|^{\alpha-1}$ behaves like $\cot(\frac{\alpha\pi}{2}) \operatorname{sgn}(x)|x|^{\alpha-1}$ near $x = 0$, for all $0 < \alpha < 1$. Once this is verified, we determine that the Hilbert transform of a function $h(x) \sim \cot(\frac{\alpha\pi}{2}) \operatorname{sgn}(x)|x|^{\alpha-1}$ satisfies $\sim -|x|^{\alpha-1}$, because $H^2 = -\operatorname{Id}$.

We now justify the aforementioned. Consider $f(x) = |x|^{-\alpha}G(x)$, where G is a smooth and decaying function with $G(0) = 1$. Thus,

$$H[f](x) = \text{P.V.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-z} |z|^{-\alpha} G(z) dz.$$

We shall show that

$$\lim_{x \rightarrow 0^+} |x|^\alpha H[f](x) = \text{const.}$$

To see this, we take small $0 < x \ll 1$ and make a change of variables $z = xw$:

$$|x|^\alpha H[f](x) = \text{P.V.} \frac{1}{\pi} \int_{w \in \mathbb{R}} \frac{1}{|w|^\alpha} \frac{1}{1-w} G(xw) dw.$$

Formally, the right-hand side goes to

$$\text{P.V.} \frac{1}{\pi} \int_{w \in \mathbb{R}} \frac{1}{|w|^\alpha} \frac{1}{1-w} dw$$

as $x \rightarrow 0$. To rigorously prove this, we take the difference and split the domain of integration as follows:

$$\begin{aligned} & \text{P.V.} \frac{1}{\pi} \int_{w \in \mathbb{R}} \frac{1}{|w|^\alpha} \frac{1}{1-w} (G(xw) - G(0)) dw \\ &= \int_{|w-1| < x^\epsilon} + \int_{x^\epsilon \leq |w-1| < 1} + \int_{1 \leq |w-1| < x^{-\epsilon}} + \int_{x^{-\epsilon} \leq |w-1|} \cdots =: \text{I} + \text{II} + \text{III} + \text{IV}, \end{aligned}$$

where $\epsilon > 0$ is a small constant. First,

$$|\text{IV}| \leq C \|G\|_{L^\infty} \int_{|w| \geq \frac{1}{2} x^{-\epsilon}} \frac{1}{|w|^{\alpha+1}} dw \leq C x^{\epsilon\alpha}.$$

Next, with $|G(xw) - G(0)| \leq \|G'\|_{L^\infty} |xw|$,

$$|\text{II}| \leq C \|G'\|_{L^\infty} \int_{x^\epsilon \leq |w-1| < 1} \frac{|x| |w|^{1-\alpha}}{|1-w|} dw \leq C x |\ln(x^\epsilon)|,$$

$$\begin{aligned}
|\text{III}| &\leq C \|G'\|_{L^\infty} \int_{1 \leq |w-1| < x^{-\epsilon}} \frac{x|w|^{1-\alpha}}{|1-w|} dw \\
&\leq C \int_{1 \leq |w-1| < x^{-\epsilon}} \frac{x}{|w|^\alpha} dw \leq C x^{1-\epsilon(1-\alpha)}.
\end{aligned}$$

Lastly, we rewrite I as follows:

$$\begin{aligned}
\text{I} &= \text{P.V.} \frac{1}{\pi} \int_{|w-1| < x^\epsilon} \frac{1}{|w|^\alpha} \frac{1}{1-w} (G(xw) - G(x)) dw \\
&\quad + \text{P.V.} \frac{1}{\pi} \int_{|w-1| < x^\epsilon} \frac{1}{|w|^\alpha} \frac{1}{1-w} (G(x) - G(0)) dw.
\end{aligned}$$

The second term is bounded by

$$\begin{aligned}
|G(x) - G(0)| \left| \text{P.V.} \frac{1}{\pi} \int_{|w-1| < x^\epsilon} \frac{1}{|w|^\alpha} \frac{1}{1-w} dw \right| \\
\leq C \|G'\|_{L^\infty} x \left| \int_{|w-1| < x^\epsilon} \left(\frac{1}{|w|^\alpha} - 1 \right) \frac{1}{1-w} dw \right|,
\end{aligned}$$

and because $x < 1$,

$$\left| \left(\frac{1}{|w|^\alpha} - 1 \right) \frac{1}{1-w} \right| \leq C, \quad |w-1| < x^\epsilon.$$

Next, with $|G(xw) - G(x)| \leq \|G'\|_{L^\infty} x |1-w|$,

$$\left| \text{P.V.} \frac{1}{\pi} \int_{|w-1| < x^\epsilon} \frac{1}{|w|^\alpha} \frac{1}{1-w} (G(xw) - G(x)) dw \right| \leq C x \int_{|w-1| < x^\epsilon} \frac{1}{|w|^\alpha} dw \leq C x^{1+\epsilon}.$$

When the estimates are collected, for small $\epsilon > 0$ depending only on α ,

$$|\text{I}| + |\text{II}| + |\text{III}| + |\text{IV}| \rightarrow 0, \quad x \rightarrow 0.$$

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